



AYDIN ADNAN MENDERES UNIVERSITY FACULTY OF ENGINEERING

Department of Electrical and Electronics Engineering

EE303 – Signals and Systems

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1. Signals & Systems (Chapter 1 of Oppenheim – 6 Hrs.)

Introduction, Continuous-Time and Discrete-Time Signals, Periodicity, Even and Odd Signals, Exponential and Sinusoidal Signals, the Unit Impulse and Unit Step Functions, Continuous-Time and Discrete-Time Systems, Basic System Properties (Memory, Invertibility, Causality, Stability, Time Invariance, Linearity)

2. Linear Time-Invariant (LTI) Systems (Chapter 2 of Oppenheim – 9 Hrs.)

Discrete-Time LTI Systems, Convolution, Continuous-Time LTI Systems, Properties of LTI Systems, LTI Systems with and without Memory, Invertibility of LTI Systems, Causality for LTI Systems, Stability for LTI Systems, Unit Step Response of an LTI System, Causal LTI Systems Described by Differential and Difference Equations, Linear Constant-Coefficient Differential Equations

3. Fourier Series (Chapter 3 of Oppenheim – 6 Hrs.)

Fourier Series Representation of CT Periodic Signals, Convergence of Fourier Series, Gibs Phenomenon, Properties of CT Fourier Series, Fourier Series Representation of DT Periodic Signals, Properties of DT Fourier Series, Fourier Series and LTI Systems

4. CT Fourier Transform (Chapter 4 of Oppenheim – 6 Hrs.)

Op-Amp Terminals & Representation of Aperiodic CT Signals, Convergence of CT Fourier Transforms, Fourier Transform for Periodic Signals, Properties of CT Fourier Transform, Convolution Property, Systems Characterized by Linear Constant-Coefficient Differential Equations

5. DT Fourier Transform (Chapter 5 of Oppenheim – 6 Hrs.)

Representation of Aperiodic DT Signals, Convergence of DT Fourier Transforms, Fourier Transform for Periodic Signals, Properties of DT Fourier Transform, Convolution Property, Systems Characterized by Linear Constant-Coefficient Differential Equations

6. The Laplace Transform (Chapter 9 of Oppenheim – 6 Hrs.)

Introduction, The Region of Convergence for Laplace Transforms, The Inverse Laplace Transform, Pole-Zero Plot, First-Order and Second-Order Systems, Properties of the Laplace Transform, Analysis and Characterization of LTI Systems Using the Laplace Transform, The Unilateral Laplace Transform

7. The Z-Transform (Chapter 10 of Oppenheim – 6 Hrs.)

Introduction, The Region of Convergence for the Z-Transforms, The Inverse Z-Transform, Pole-Zero Plot, First-Order and Second-Order Systems, Properties of the Z-Transform, Analysis and Characterization of LTI Systems Using the Z-Transform, The Unilateral Z-Transform

outline

Ch3: Fourier Series

Introduction

- In this chapter, we focus on the representation of continuous-time and discrete-time **periodic signals** referred to as the **Fourier series**.

- In Chapters 4 and 5, we extend the analysis to the **Fourier transform** representation of broad classes of aperiodic, finite energy signals.

- Together, these representations provide one of the most powerful and important sets of tools and insights for analyzing, designing, and understanding signals and LTI systems, and we devote considerable attention in this and subsequent chapters to exploring the uses of Fourier methods.



Figure 3.2 Jean Baptiste Joseph Fourier [picture from J. B. J. Fourier, *Oeuvres de Fourier*, Vol. II (Paris: Gauthier-Villars et Fils, 1980)].

Ch3: Fourier Series

Introduction

- The representation and analysis of LTI systems through the convolution sum as developed in Chapter 2 is based on representing signals as linear combinations of shifted impulses.
- In this and the following two chapters, we explore an alternative representation for signals and LTI systems.
- As in Chapter 2, the starting point for our discussion is the development of a representation of signals as linear combinations of a set of basic signals. For this alternative representation we use **complex exponentials**.
- The resulting representations are known as the **continuous-time and discrete-time Fourier series and transform**.
- As we will see, these can be used to construct broad and useful classes of signals.

Ch3: Fourier Series

Historical Background (Reading)

- The development of Fourier analysis has a long history involving a great many individuals and the investigation of many different physical phenomena. The concept of using "trigonometric sums"–that is, sums of harmonically related sines and cosines or periodic complex exponentials–to describe periodic phenomena goes back at least as far as the Babylonians, who used ideas of this type in order to predict astronomical events.
- The modern history of the subject begins in 1748 with **L. Euler**, who examined the motion of a vibrating string.

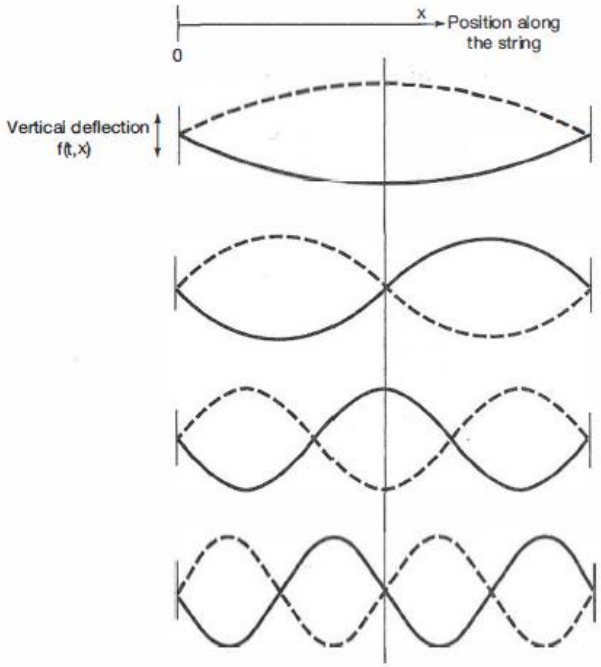


Figure 3.1 Normal modes of a vibrating string. (Solid lines indicate the configuration of each of these modes at some fixed instant of time, t.)



Leonhard Euler (1707 – 1783) was a Swiss mathematician, physicist, astronomer, logician and engineer, who made important and influential discoveries in many branches of mathematics, such as infinitesimal calculus and graph theory, while also making pioneering contributions to several branches such as topology and analytic number theory.

Ch3: Fourier Series

Historical Background (Reading)

- What Euler noted was that if the configuration of a vibrating string at some point in time is a **linear combination of these normal modes**, so is the configuration at any subsequent time. Furthermore, Euler showed that one could calculate the coefficients for the linear combination at the later time in a very straightforward manner from the coefficients at the earlier time.

- In doing this, Euler performed the same type of calculation as we will in the next section in deriving one of the properties of trigonometric sums that make them so useful for the analysis of LTI systems.

- Specifically, we will see that if the input to an LTI system is expressed as a linear combination of periodic complex exponentials or sinusoids, the output can also be expressed in this form, with coefficients that are related in a straightforward way to those of the input.

- The property described in the preceding paragraph would not be particularly useful, unless it were true that a **large class of interesting functions** could be represented by linear combinations of complex exponentials.

Ch3: Fourier Series

Historical Background (Reading)

- In 1753, D. Bernoulli argued on physical grounds that all physical motions of a string could be represented by linear combinations of normal modes, but he did **not pursue this mathematically**, and his ideas were not widely accepted.

- In fact, Euler himself discarded trigonometric series, and in 1759 J. L. Lagrange strongly criticized the use of trigonometric series in the examination of vibrating strings. His criticism was based on his own belief that it was impossible to represent signals with corners (i.e., with discontinuous slopes) using trigonometric series.

- By 1807, Fourier had completed a work, Fourier had found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body. In addition, **he claimed that "any" periodic signal could be represented by such a series.** While his treatment of this topic was significant, many of the basic ideas behind it had been discovered by others.

- Also, Fourier's mathematical arguments were still imprecise, and it remained for P. L. Drichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a Fourier series.

Ch3: Fourier Series

Historical Background (Reading)

- Four distinguished mathematicians and scientists were appointed to examine the 1807 paper of Fourier.

- Three of the four–S. F. Lacroix, G. Monge, and P. S. de Laplace–were in favor of publication of the paper, but the fourth, J. L. Lagrange, remained adamant in rejecting trigonometric series, as he had done 50 years earlier.

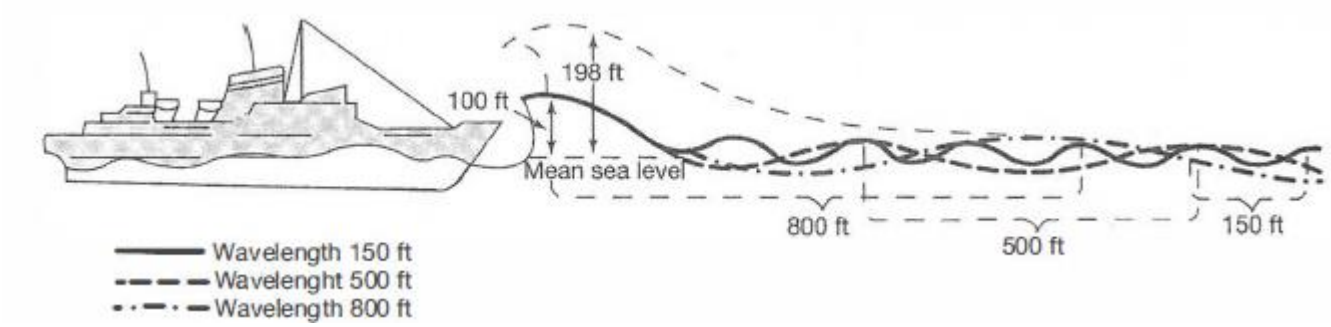
- Because of Lagrange's vehement objections, Fourier's paper never appeared. After several other attempts to have his work accepted and published by the Institut de France, Fourier undertook the writing of another version of his work, which appeared as the text *Theorie analytique de la chaleur.*

- This book was published in 1822, 15 years after Fourier had first presented his results to the Institut.

Ch3: Fourier Series

Historical Background (Reading)

- In addition to the original studies of vibration and heat diffusion, there are numerous other problems in science and engineering in which sinusoidal signals, and therefore Fourier series and transforms play an important role. For example, sinusoidal signals arise naturally in describing the **motion of the planets** and the **periodic behavior of the earth's climate**. **Alternating-current sources** generate sinusoidal voltages and currents, and, as we will see, the tools of Fourier analysis enable us to **analyze the response of an LTI system**, such as a circuit, to such sinusoidal inputs.
- Also, as illustrated in Figure below, **waves in the ocean** consist of the linear combination of sinusoidal waves with different spatial periods or *wavelengths.* Signals **transmitted by radio and television stations** are sinusoidal in nature as well, and as a quick perusal of any text on Fourier analysis will show, the range of applications in which sinusoidal signals arise and in which the tools of Fourier analysis are useful extends far beyond these few examples.



Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

THE RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

- It is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:
  - 1. The set of basic signals can be used to construct a broad and useful class of signals.
  - 2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

- The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is,

$$\begin{aligned} &\textit{Continuous time}: && e^{st} \rightarrow H(s)e^{st} \\ &\textit{Discrete time}: && z^{n} \rightarrow H(z)z^{n} \end{aligned}$$

where the complex amplitude factor $\mathbf{H(s)}$ or $\mathbf{H(z)}$ will in general be a function of the complex variable **'s'** or '**z**'.

- A signal for which the system output is a (possibly complex) constant times the input is referred to as an **eigenfunction** of the system, and the amplitude factor is referred to as the system's **eigenvalue**.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

THE RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

- A signal for which the system output is a (possibly complex) constant times the input is referred to as an **eigenfunction** of the system, and the amplitude factor is referred to as the system's **eigenvalue**.

$$\begin{aligned} Continuous\ time: &\quad e^{st} \rightarrow H(s)e^{st} \\ Discrete\ time: &\quad z^n \rightarrow H(z)z^n \end{aligned}$$

- Hence, the complex exponentials are eigenfunctions of LTI systems.

- The constant $\mathbf{H(s)}$ (or $\mathbf{H(z)}$) for a specific value of 's' (or 'z') is then the eigenvalue associated with the eigenfunction e^{st} (z^n).

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \qquad H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Fourier Series Representation of CT Periodic Signals

THE RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

- For the analysis of LTI systems, the usefulness of decomposing more general signals in terms of eigenfunctions can be seen from an example.
- Let x(t) corresponds to a linear combination of three complex exponentials; that is,

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

- From the eigenfunction property, the response to each separately is

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t},$$
$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t},$$
$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t},$$

- and from the superposition property the response to the sum is the sum of the responses, so that

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

- Specifically, if the input to a continuous-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x(t) = \sum_k a_k e^{s_k t} \quad \Rightarrow \quad y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

- A signal is periodic if, for some positive value of ***T,***

$$x(t) = x(t+T) \quad ; \quad \forall t$$

- The **fundamental period** of x(t) is the minimum positive, nonzero value of **T** for which eq. above is satisfied, and the value $\omega_0 = 2\pi/T$ is referred to as the **fundamental frequency**.

- Two basic periodic signals are; the sinusoidal signal $x(t) = \cos(w_0 t)$

and the periodic complex exponential $x(t) = e^{jw_0 t}$

- Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$.

- The set of *harmonically related* complex exponentials is;

$$\emptyset_k(t) = e^{jkw_0 t} = e^{jk(2\pi/T)t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

$$\emptyset_k(t) = e^{jkw_0 t} = e^{jk(2\pi/T)t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- Each of these signals has a fundamental frequency that is a multiple of ω_0, and therefore, each is periodic with period T. Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

  is also periodic with period T.

- The components for $k = +N$ and $k = -N$ are referred to as the ***N*th** harmonic components.

- The representation of a periodic signal in the form of eq. above is referred to as the ***Fourier series*** **representation**. Before developing the properties of this representation, let us consider an example.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.2 from text book)

Consider a periodic signal $x(t)$, with fundamental frequency 2π,

$$x(t) = \sum_{k=-3}^{3} a_k e^{jkw_0 t} = \sum_{k=-3}^{3} a_k e^{jk2\pi t} \quad \text{where} \quad \begin{aligned} a_0 &= 1, \\ a_1 = a_{-1} &= 1/4, \\ a_2 = a_{-2} &= 1/2, \\ a_3 = a_{-3} &= 1/3, \end{aligned}$$

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t} = a_{-3}\, e^{j(-3)2\pi t} + a_{-2}\, e^{j(-2)2\pi t} + a_{-1}\, e^{j(-1)2\pi t} + a_0\, e^{j(0)2\pi t} + a_3 e^{j(3)2\pi t} + a_2\, e^{j(2)2\pi t} + a_1\, e^{j(1)2\pi t}$$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

Using Euler's relation, $x(t)$can be expressed in the form;

$$x(t) = 1 + \frac{1}{2}cos(2\pi t) + cos(4\pi t) + \frac{2}{3}\left(cos(6\pi t)\right)$$

Euler's formulas

$$\left(e^{j\theta} + e^{-j\theta}\right) = 2\cos\theta$$

$$\left(e^{j\theta} - e^{-j\theta}\right) = 2\mathrm{j}\sin\theta$$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.2 from text book)

$$x(t) = 1 + \frac{1}{2}cos(2\pi t) + cos(4\pi t) + \frac{2}{3}\left(cos(6\pi t)\right)$$

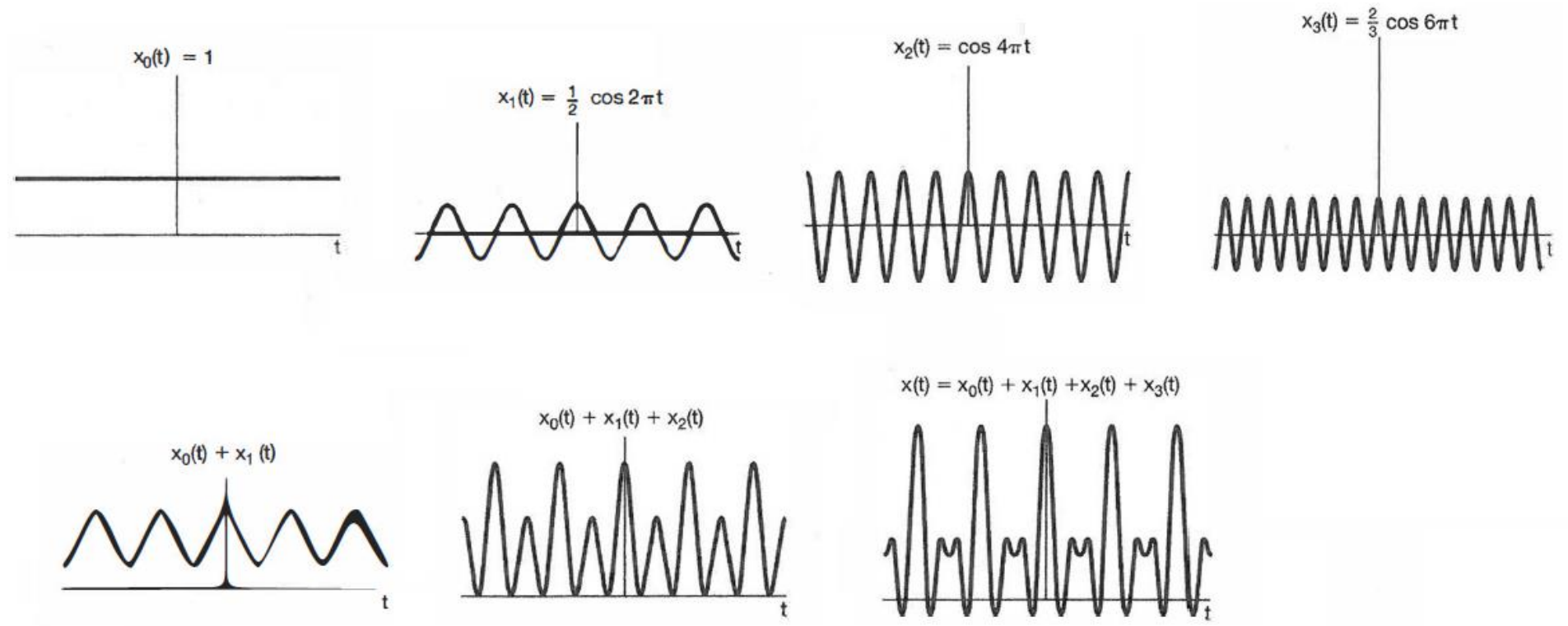


Figure 3.4 Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

- Specifically, suppose that $x(t)$ is real; $x^*(t) = x(t)$, we obtain

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}.$$

Replacing k by $-k$ in the summation, we have $x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$, which requires that $a_k = a_{-k}^*$,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t} \Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}] \Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}].$$

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{a_k e^{jk\omega_0 t}\}.$$

If a_k is expressed in polar form as $a_k = A_k e^{j\theta_k}$,

then eq. becomes $x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{A_k e^{j(k\omega_0 t + \theta_k)}\}.$

That is, $x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k).$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k).$$

- This equation is one commonly encountered form for the Fourier series of real periodic signals in continuous time. Another form is obtained by writing a_k in rectangular form as;

$$a_k = B_k + jC_k,$$

where B_k and C_k are both real. With this expression for a_k, eq. above takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t].$$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

We can find the a_k's easily? But HOW? **Prove It !!!!**

Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

Analysis equation

$$a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_{<T>} x(t)e^{-jk(2\pi/T)t}dt$$

Fourier Series Pair

Fourier Series coefficients

The coefficient a_0 is the dc or constant component of $x(t)$ with $k = 0$. That is,

$a_0 = \frac{1}{T}\int_T x(t)dt$, which is simply the average value of $x(t)$ over one period.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.3 from text book)

Consider the signal;

$$x(t) = sin(w_0 t)$$

Find the fourier series coefficients?

Solution:

We can directly apply FS analysis equation for $x(t)$ $\rightarrow a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jkw_0 t}dt$

Or **THINK**. What is the meaning of Fourier Series representation?

Using Euler's relation, $x(t)$can be expressed in the form;

$$x(t) = sin(w_0 t) = \frac{1}{2j}e^{jw_0 t} - \frac{1}{2j}e^{-jw_0 t}$$

Euler's formulas

$$\left(e^{j\theta} + e^{-j\theta}\right) = 2\cos\theta$$

$$\left(e^{j\theta} - e^{-j\theta}\right) = 2\mathrm{j}\sin\theta$$

Now $x(t)$ in desired form of $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.3 from text book)

Consider the signal;

$$x(t) = sin(w_0 t)$$

Find the fourier series coefficients?

Solution:

$$x(t) = sin(w_0 t) = \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \cdots + a_{-2}\, e^{j(-2)w_0 t} + a_{-1}\, e^{j(-1)w_0 t} + a_0\, e^{j(0)w_0 t} + a_1\, e^{j(1)w_0 t} + \cdots$$

İn our case, we have only a_1 and a_{-1}

$k = 1$ $\qquad$ $k = -1$

$$x(t) = sin(w_0 t) = \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t}$$

$$a_1 = \frac{1}{2j} \quad a_{-1} = -\frac{1}{2j}$$

$$a_k = 0, k \neq \pm 1$$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.4 from text book)

Let

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Find the fourier series coefficients?

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$

$$a_k = 0, \quad |k| > 2.$$

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.4 from text book)

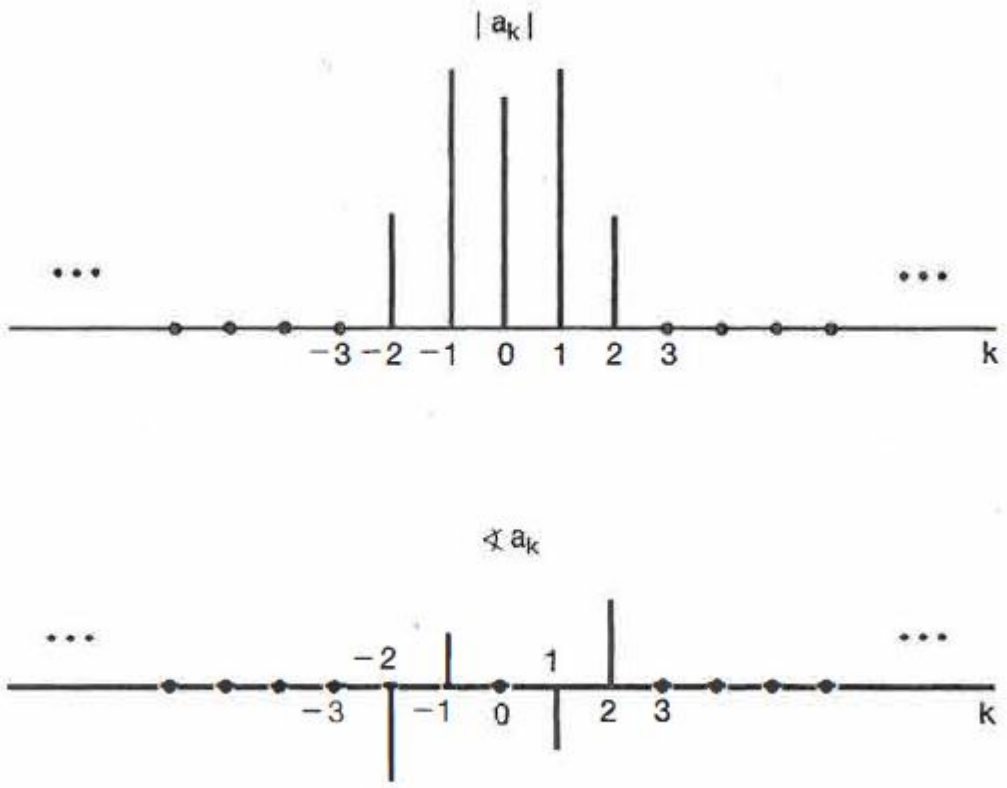


Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.5 from text book)

Let $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$, Find the fourier series coefficients?

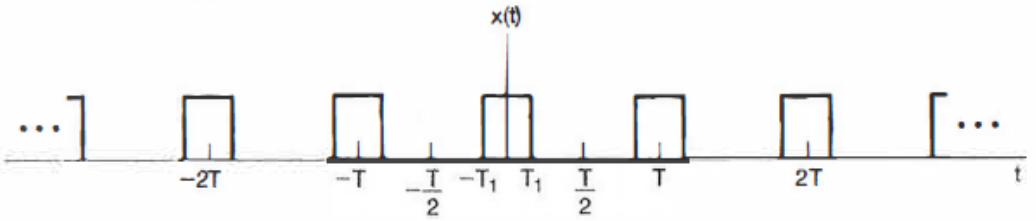


Figure 3.6 Periodic square wave.

we have first, for $k = 0$, $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$.

For $k \neq 0$, we obtain $a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$, $a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$.

$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$, $k \neq 0$, where we have used the fact that $\omega_0 T = 2\pi$.

For $T = 4T_1$, $\omega_0 T_1 = \pi/2$, $a_k = \frac{\sin(\pi k/2)}{k\pi}$, $k \neq 0$, $a_0 = \frac{1}{2}$.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Ex: (3.5 from text book)

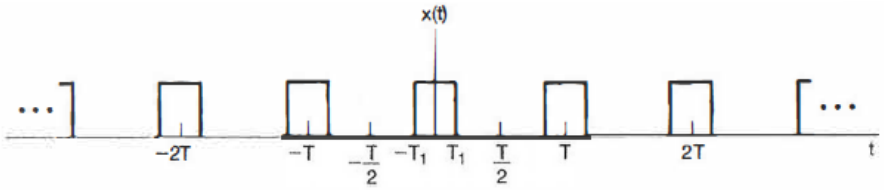
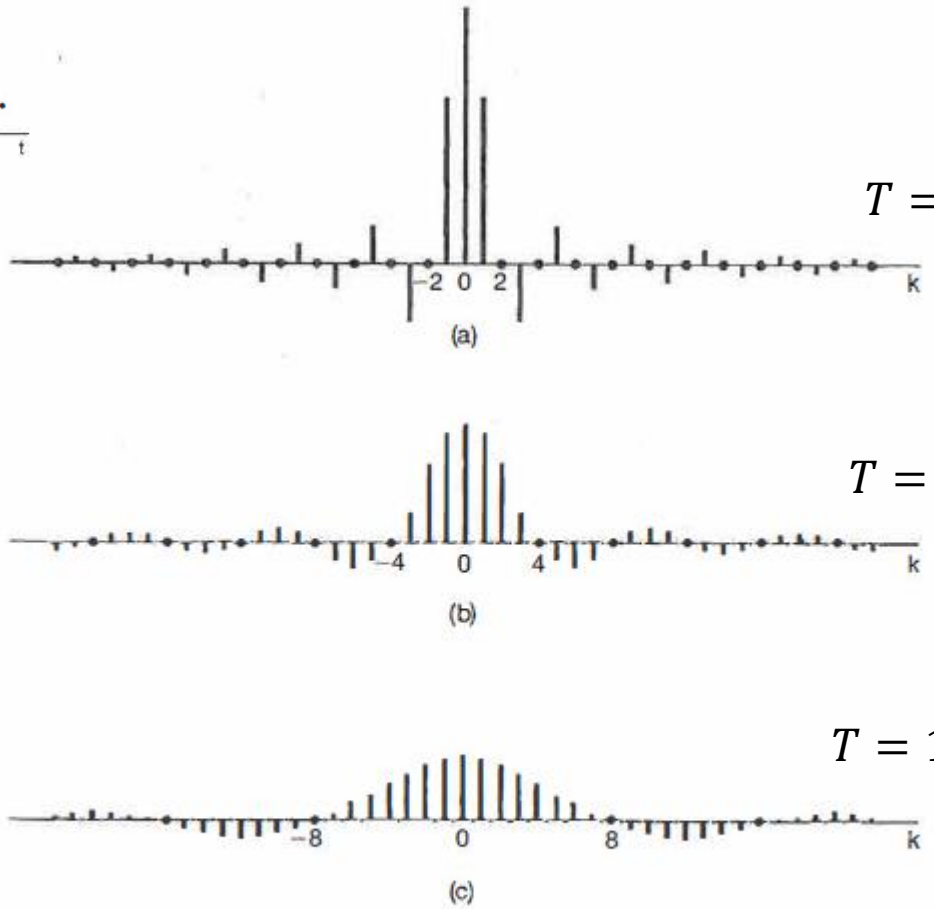


Figure 3.6 Periodic square wave.

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0,$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$



$T = 4T_1$

$T = 8T_1$

$T = 16T_1$

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Ch3: Fourier Series

Convergence of the Fourier Series

- Although **Euler** and **Lagrange** would have been happy with the results of Examples 3.3 and 3.4, they would have objected to Example 3.5, since **x(t) is discontinuous** while each of **its harmonic components is continuous**.

- **Fourier**, on the other hand, considered the same example and maintained that the Fourier series representation of the square wave is valid.

- In fact, Fourier maintained that **any periodic signal** could be represented by a Fourier series.

- Although **this is not quite true**, it is true that Fourier series can be used to represent an **extremely large class of periodic signals**, including the square wave and all other periodic signals with which we will be concerned in this course and which are of interest in practice.

Ch3: Fourier Series

Convergence of the Fourier Series

- Let us turn now to the question of when a periodic signal x(t) does in fact have a Fourier series representation. Of course, for any signal, we can attempt to obtain a set of Fourier coefficients through the use of analysis equation. However, in some cases, the integral in a_k may diverge; that is, the value obtained for some of the a_k may be infinite.

- Moreover, even if all of the coefficients obtained from analysis equation are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.

- Fortunately, there are no convergence difficulties for large classes of periodic signals.

- One class of periodic signals that are representable through the Fourier series is those signals which have **finite energy** over a single period, i.e., signals for which

$$\int_{<T>} |x(t)|^2 dt < \infty$$

Ch3: Fourier Series

Convergence of the Fourier Series – Dirichlet conditions

- Moreover, an alternative set of conditions, developed by **P. L. Dirichlet** and also satisfied by essentially all of the signals with which we will be concerned, **guarantees that $x(t)$ *equals* its Fourier series representation**, except at isolated values *of 't'* for which $x(t)$ is discontinuous. At these values, the infinite series of equation given below converges to the average of the values on either side of the discontinuity.

$$\sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

- The **Dirichlet conditions** are as follows:

Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_{<T>} |x(t)|dt < \infty$$

Ch3: Fourier Series

Convergence of the Fourier Series – Dirichlet conditions

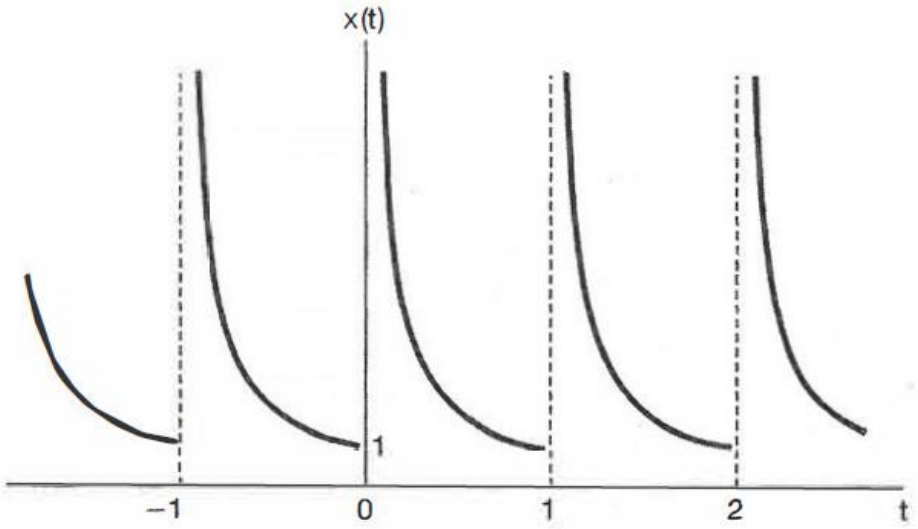
Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_{<T>} |x(t)|dt < \infty$$

A periodic signal that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1;$$

that is, $x(t)$ is periodic with period 1.



Ch3: Fourier Series

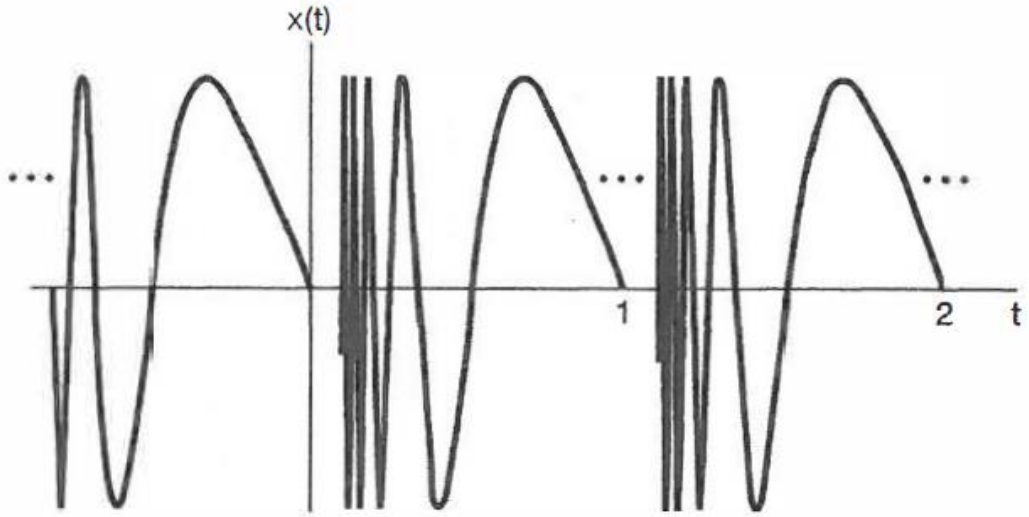
Convergence of the Fourier Series – Dirichlet conditions

Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

An example of a function that meets Condition 1 but not Condition 2 is

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1,$$

as illustrated in Figure. For this function, which is periodic with $T = 1$, $\int_0^1 |x(t)|\, dt < 1$.

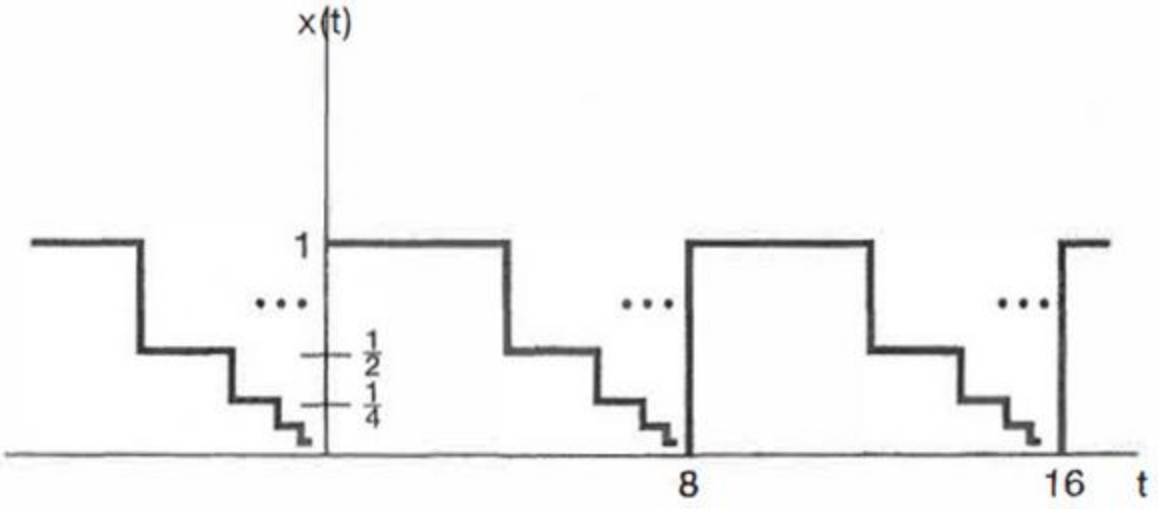


The function has, however, an infinite number of maxima and minima in the interval.

Ch3: Fourier Series

Convergence of the Fourier Series – Dirichlet conditions

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.



a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \le t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \le t < 4$, $x(t) = 1/2$, $4 \le t < 6$, $x(t) = 1/4$, $6 \le t < 7$, $x(t) = 1/8$. $7 \le t < 7.5$. etc.].

Ch3: Fourier Series

Convergence of the Fourier Series – Dirichlet conditions

- As can be seen from the examples given in previous figures, signals that do not satisfy the Dirichlet conditions are generally pathological in nature and consequently do not typically arise in practical contexts.

- For this reason, the question of the convergence of Fourier series will not play a particularly significant role in the remainder of the course.

- For a **periodic signal that has no discontinuities**, the Fourier series representation converges and equals the original signal at every value of 't'.

- For a **periodic signal with a finite number of discontinuities** in each period, the Fourier series representation equals the signal everywhere except at the isolated points of discontinuity, at which the series converges to the average value of the signal on either side of the discontinuity. In this case the difference between the original signal and its Fourier series representation contains no energy, and consequently, the two signals **can be thought of as being the same** for all practical purposes.
- Specifically, since the signals differ only at isolated points, the integrals of both signals over any interval are **identical**. For this reason, the two signals behave identically under convolution and consequently are identical from the standpoint of the analysis of LTI systems.

Ch3: Fourier Series

Convergence of the Fourier Series – Gibbs Phenomenon

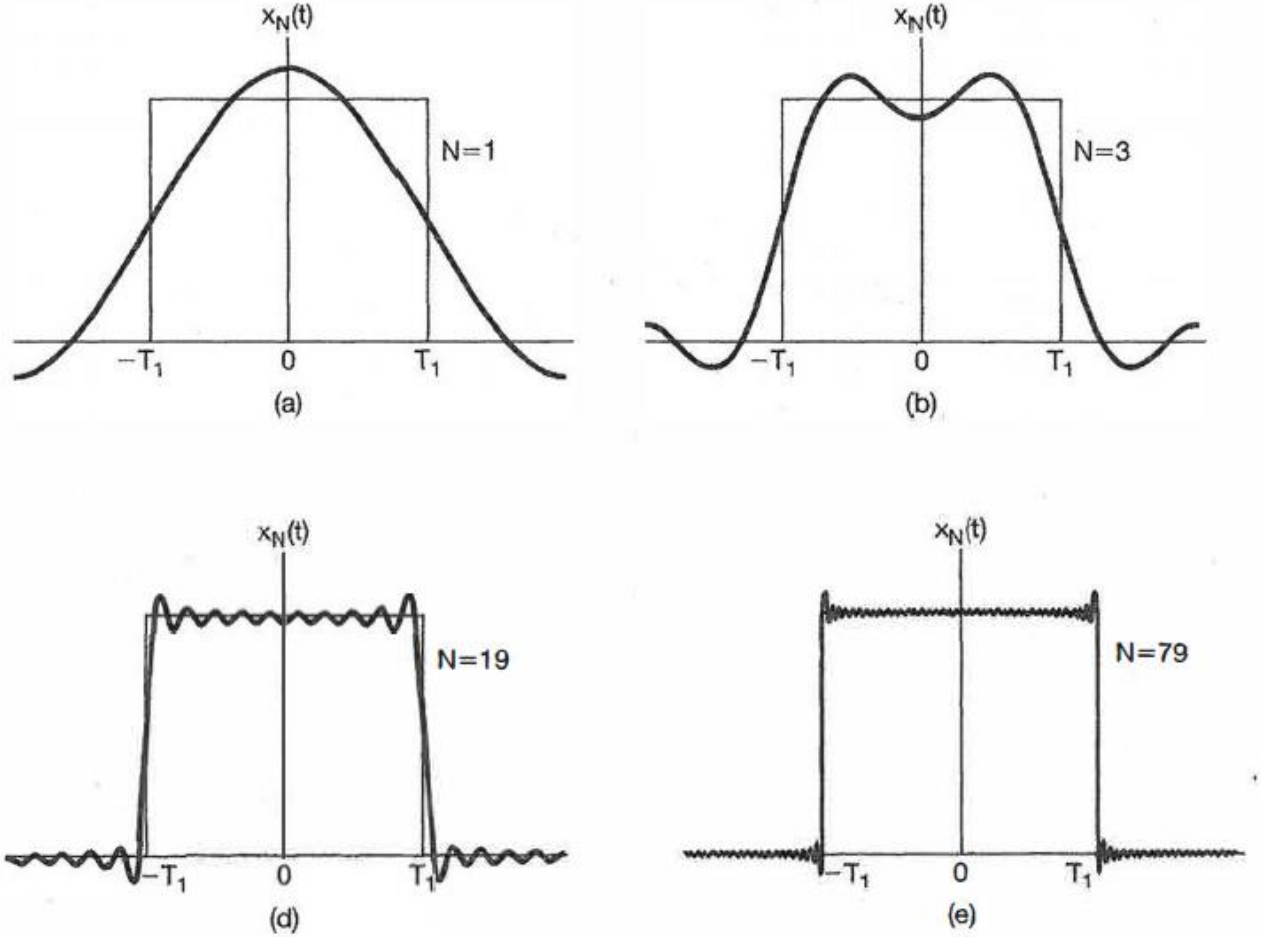


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

Ch3: Fourier Series

Fourier Series Representation of CT Periodic Signals

Summary of last week

Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

Analysis equation

$$a_k = \frac{1}{T} \int_{<T>} x(t) e^{-jkw_0 t} dt = \frac{1}{T} \int_{<T>} x(t) e^{-jk(2\pi/T)t} dt$$

Fourier Series Pair

Fourier Series coefficients

Convergence of the Fourier Series

The **Dirichlet conditions**

$$\int_{<T>} |x(t)|^2 dt < \infty$$

$$\int_{<T>} |x(t)| dt < \infty$$

Ch3: Fourier Series

Properties of the CT Fourier Series

- Suppose that *x(t)* is a periodic signal with period *T* and fundamental frequency $w_0 = 2\pi/T$.
- Then if the Fourier series coefficients of *x(t)* are denoted by a_k, we will use the notation;

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

1. Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, respectively. That is,

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

$$z(t) = Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k.$$

The proof of this follows directly from the application of analysis eq.

Ch3: Fourier Series

Properties of the CT Fourier Series

2. Time Shift

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

Proof:

When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved. The Fourier series coefficients b_k of the resulting signal $y(t) = x(t - to)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_0 t}dt.$$

Letting $\tau = t - to$ in the integral, and noting that the new variable τ will also range over an interval of duration T, we obtain

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau = e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k$$

3. Time Reversal

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$$

Proof:

The period T of a periodic signal $x(t)$ also remains unchanged when the signal undergoes time reversal.

To determine the Fourier series coefficients of $y(t) = x(-t)$, let us consider the effect of time reversal on the synthesis equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}$$

Making the substitution $k = -m$, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

We observe that the right-hand side of this equation has the form of a Fourier series synthesis equation for $x(-t)$, where the Fourier series coefficients b_k are

$$b_k = a_{-k}$$

4. Time Scaling

- Time scaling is an operation that in general changes the period of the underlying signal.
- Specifically, if $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.
- Since the time-scaling operation applies directly to each of the harmonic components of $x(t)$, we may easily conclude that the Fourier coefficients for each of those components remain the same.
- That is, if $x(t)$ has the Fourier series representation in Synthesis eq., then

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha w_0)t}$$

is the Fourier series representation of $x(\alpha t)$.
We emphasize that, while the Fourier coefficients have not changed, the Fourier series representation has changed because of the change in the fundamental frequency.

Ch3: Fourier Series

Properties of the CT Fourier Series

5. Multiplication

Suppose that $x(t)$ and $y(t)$ are both periodic with period T and that

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, \qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for $x(t)$ and $y(t)$. The result is

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

6. Conjugation and Conjugate Symmetry

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}^*.$$

This property is easily proved by applying complex conjugation to both sides of Synthesis eq. and replacing the summation variable k by its negative.

Ch3: Fourier Series

Properties of the CT Fourier Series

7. Parseval's Relation

Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2,$$

where the a_k are the Fourier series coefficients of $x(t)$ and T is the period of the signal.

Note that the left-hand side of eq. is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$. Also,

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2,$$

so that $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$. Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Ch3: Fourier Series

Properties of the CT Fourier Series

Table 1: **Properties of the Continuous-Time Fourier Series**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

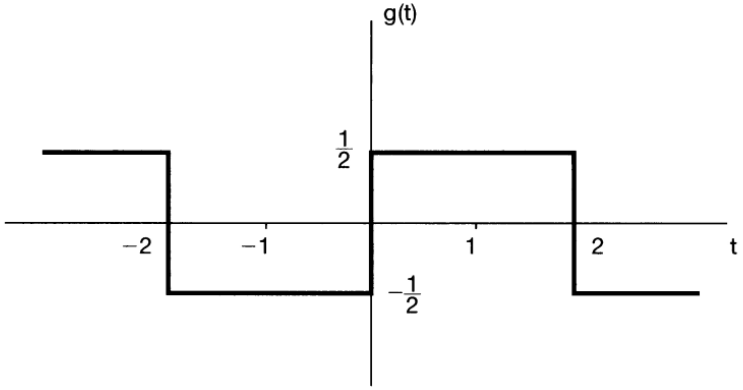
Property	Periodic Signal	Fourier Series Coefficients
	$x(t)$, $y(t)$ } Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency-Shifting	$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite-valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$a_k = a^*_{-k}$ $\Re e\{a_k\} = \Re e\{a_{-k}\}$ $\Im m\{a_k\} = -\Im m\{a_{-k}\}$ $\lvert a_k\rvert = \lvert a_{-k}\rvert$ $\measuredangle a_k = -\measuredangle a_{-k}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

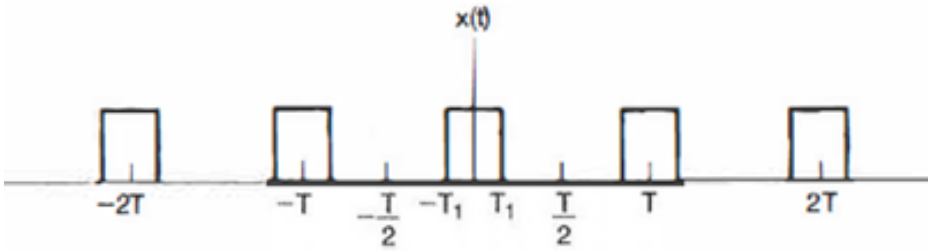
Ex: (3.6 from text book)

Consider the signal g(t) with a fundamental period of 4



We could determine the Fourier series representation of g(t) directly from the analysis equation.

Instead, we will use the relationship of g(t) to the symmetric periodic square wave x(t) in Example 3.5.



By choosing proper values for 'T' and 'T_1'

a) Express g(t) in terms of x(t).

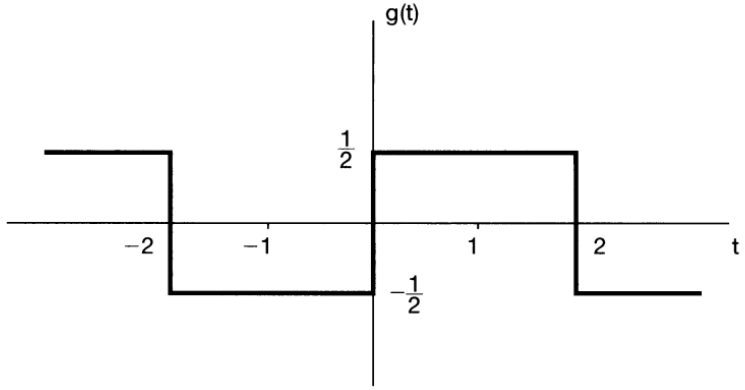
b) For your solution, T=?, T_1 =?

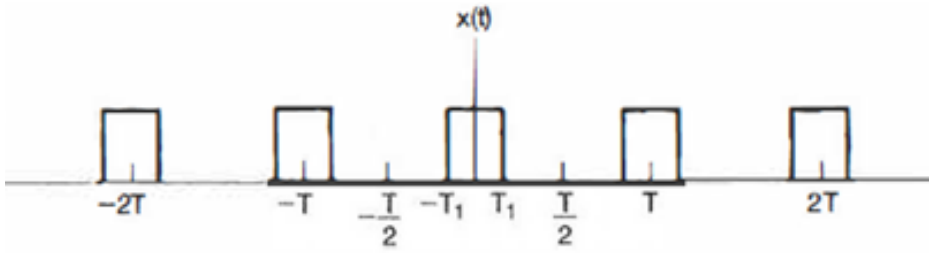
Ch3: Fourier Series

CT Fourier Series

Ex: (3.6 from text book)

Consider the signal g(t) with a fundamental period of 4





We could determine the Fourier series representation of g(t) directly from the analysis equation.

Instead, we will use the relationship of g(t) to the symmetric periodic square wave x(t) in Example 3.5.

Referring to that example, we see that, with T = 4 and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2$$

Let's remember the time-shift property in properties table:

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

Then, if the Fourier Series coefficients of $x(t)$ are denoted by a_k the Fourier coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2}.$$

Ex: (3.6 from text book)

$g(t) = x(t-1) - 1/2$

The Fourier coefficients of the dc offset in g(t) i.e., the term -1/2 are given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}.$$

Applying the linearity property, we conclude that the coefficients for g(t) may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

where each a_k may now be replaced by the corresponding expression from ex. 3.5

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0, \qquad a_0 = \frac{1}{2}.$$

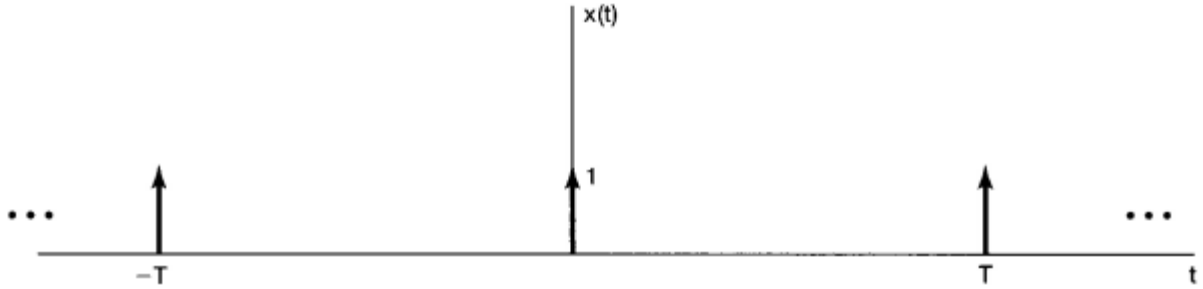
$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}.$$

Ex: (3.8 from text book)

Let us examine some properties of the Fourier series representation of a periodic train of impulses, or impulse train.

This signal and its representation in terms of complex exponentials will play an important role when we discuss the topic of sampling

The impulse train with period T may be expressed as $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

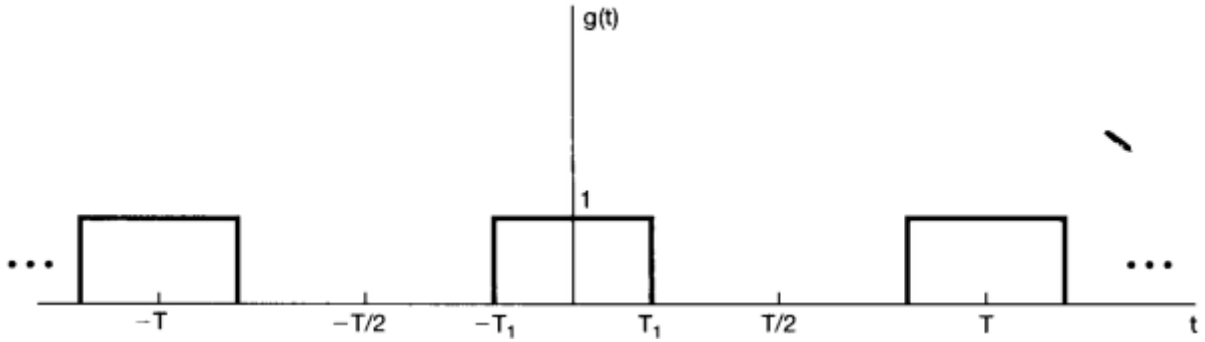


To determine the Fourier series coefficients a_k, we use analysis equation and select the interval of integration to be $-T/2 \leq t \leq T/2$, avoiding the placement of impulses at the integration limits.

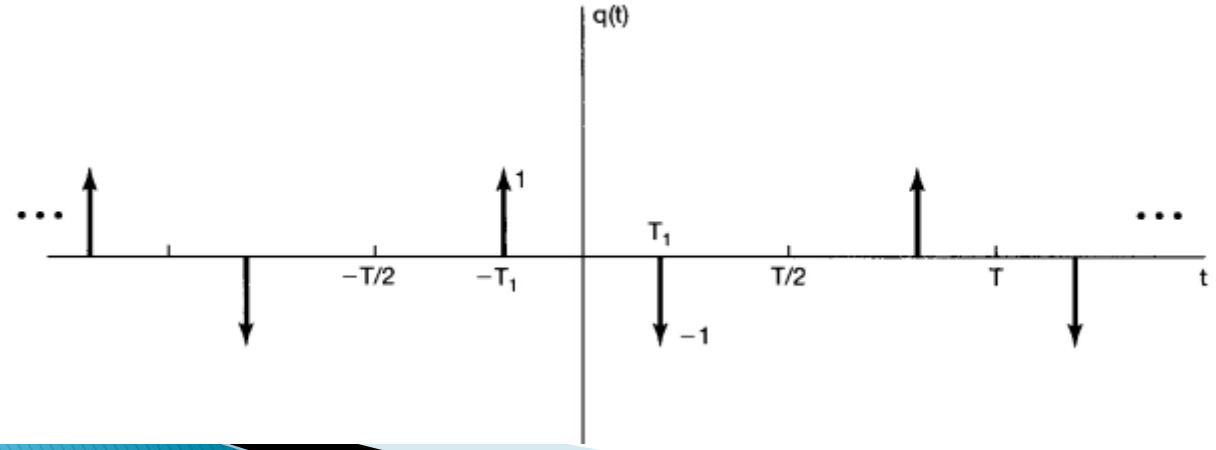
Within this interval, $x(t)$ is the same as $\delta(t)$, and it follows that $a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T}$

Ex: (3.8 from text book)

As you can see, all the Fourier series coefficients of the impulse train are identical.
These coefficients are also real valued and even (with respect to the index k).
This is to be expected, since, according to properties table, any real and even signal (such as our impulse train) should have real and even Fourier coefficients.
The impulse train also has a straightforward relationship to square-wave signals such as g(t)



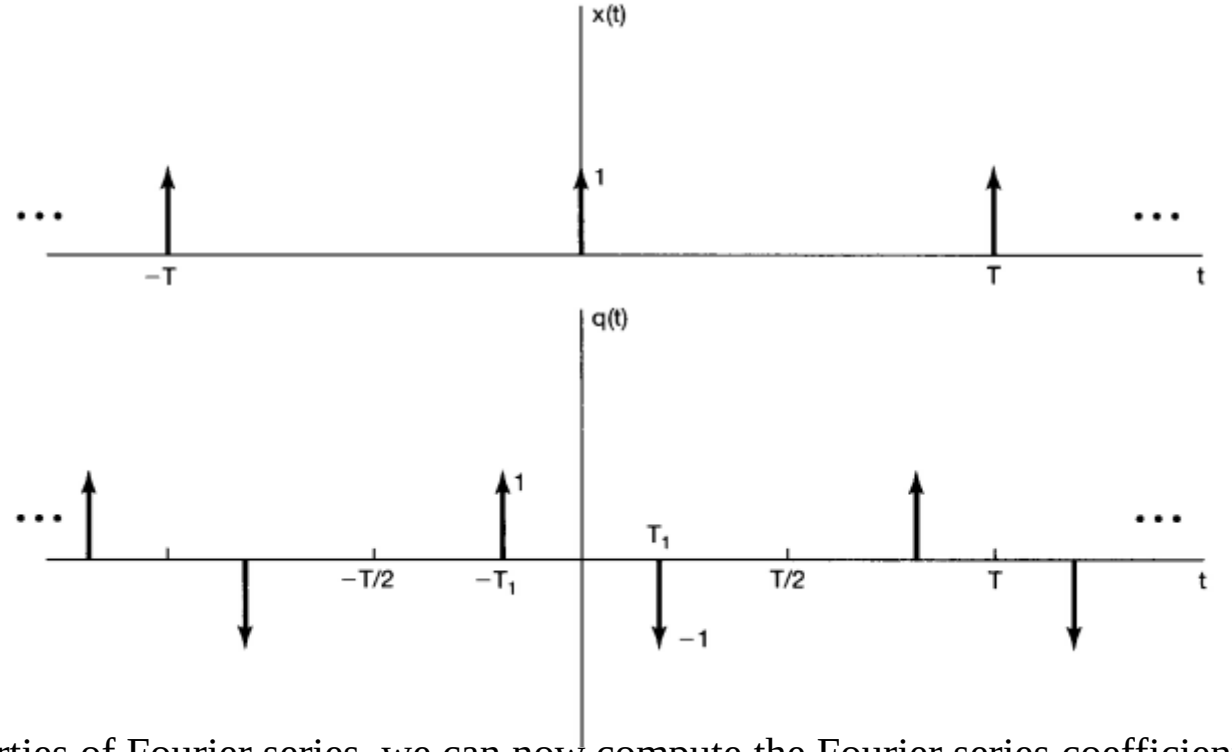
The derivative of g(t) is the signal q(t), and it is illustrated below



Ex: (3.8 from text book)

We may interpret q(t) as the difference of two shifted versions of the impulse train x(t). That is,

$$q(t) = x(t + T_1) - x(t - T_1)$$



Using the properties of Fourier series, we can now compute the Fourier series coefficients of q(t) and g(t) without any further direct evaluation of the Fourier series analysis equation.

Ex: (3.8 from text book)

First, from the time-shifting and linearity properties, we see from equation of relationship between q(t) and x(t) that

$$q(t) = x(t + T_1) - x(t - T_1)$$

the Fourier series coefficients b_k of q(t) may be expressed in terms of the Fourier series coefficients a_k of x(t);

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

Here $\omega_0 = \frac{2\pi}{T}$. If we substitute the value we calculated earlier for a_k in the above equation,

$$b_k = \frac{1}{T}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] = \frac{2j\sin(k\omega_0 T_1)}{T}$$

Finally, since q(t) is the derivative of g(t), we can use the differentiation property

$$\frac{dx(t)}{dt} \rightarrow jkw_0 a_k$$

$$b_k = jk\omega_0 c_k$$

where the c_k are the Fourier series coefficients of g(t). Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

Note that the above eq. is valid for $k \neq 0$. However, since c_0 is just the average value of g(t) over one period, we can determine it by inspection from its sketch:

$$c_0 = \frac{2T_1}{T}$$

Ch3: Fourier Series

Discrete-Time Fourier Series (DTFS)

- ✓ Up to this point, we dealed with FS for Continuous-Time signals. **(CTFS)**
- ✓ Now, we will analyze FS for Discrete-Time signals **(DTFS)**
- ✓ **CTFS & DTFS** are very similar, hence only main differences will be given by examples

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_{<T>} x(t)e^{-jk(2\pi/T)t}dt$$

CTFS Pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$$

DTFS Pair

Ch3: Fourier Series

Discrete-Time Fourier Series (DTFS)

DTFS Pair

Synthesis equation

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>}^{\infty} a_k e^{jk(2\pi/N)n}$$

Analysis equation

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

- ✓ These equations play the same role for DT periodic signals as the role of CTFS pair equations in CT case.
- ✓ As in CT, the DTFS coefficients a_k are often referred to as the spectral coefficients of $x[n]$.
- ✓ These coefficients specify a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials.

Ch3: Fourier Series

Properties of the DT Fourier Series

Table 2: **Properties of the Discrete-Time Fourier Series**

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

Property	Periodic signal	Fourier series coefficients
	$x[n]$, $y[n]$ } Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	a_k, b_k } Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shift	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shift	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r] y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)}) a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite-valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Ex: (3.10 from text book)

Consider the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$ of Example 3.3

$$x[n] = \sin \omega_0 n$$

$x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers.
For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N}$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case.
Expanding the signal as a sum of two complex exponentials by using Euler's formulas, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$

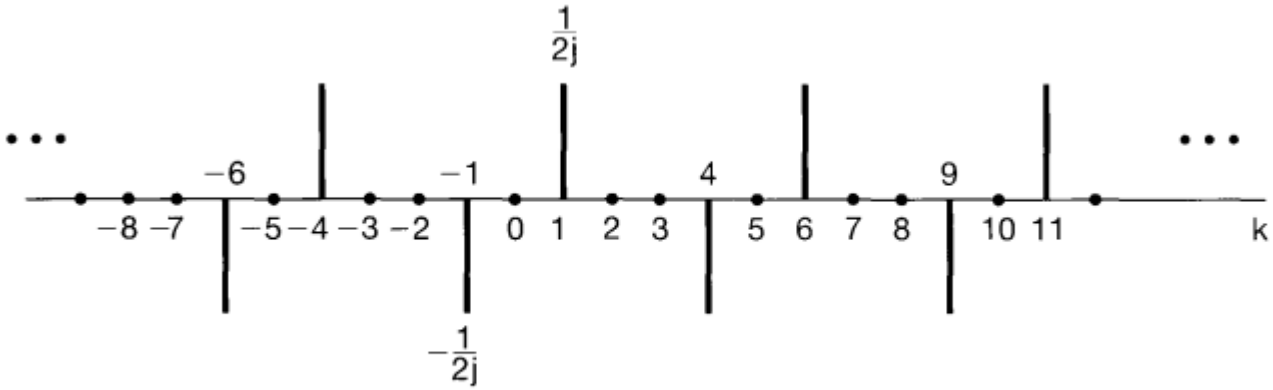
$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}$$

Remaining coefficients over the interval of summation, N are zero. These coefficients repeat with period N; thus, a_{N+1} is also equal to (1/2j) and a_{N-1} equals (-1/2j).

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

Ex: (3.10 from text book)

The Fourier series coefficients for this example with $N = 5$ are illustrated in below figure. The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (3.94).



Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Ex: (3.10 from text book)

Consider now the case when $2\pi/\omega_0$ is a ratio of integers

$$\omega_0 = \frac{2\pi M}{N}$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n}$$

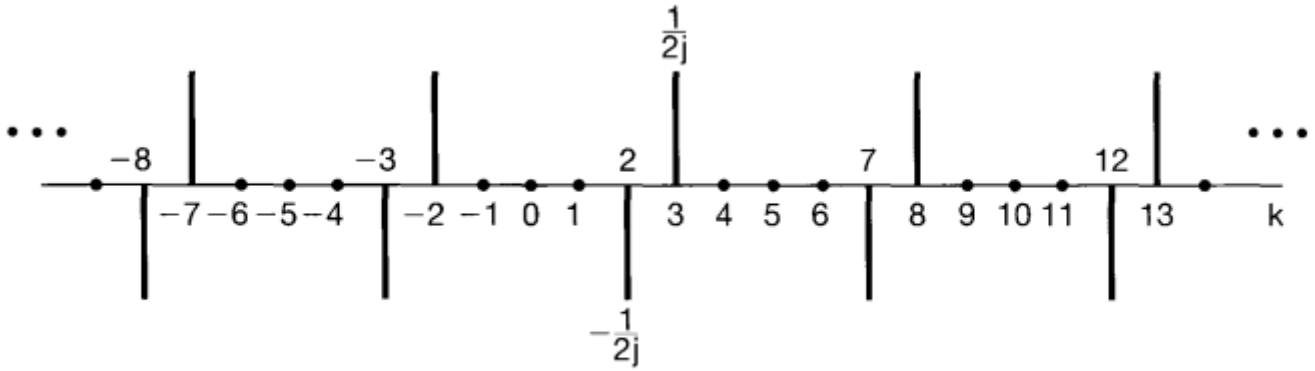
from which we can determine by inspection that $a_M = 1/2j$, $a_{-M} = -1/2j$, and the remaining coefficients over one period of length N are zero.

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$$

Ex: (3.10 from text book)

The Fourier series coefficients for this example with M=3 and N = 5 are illustrated in below figure.



Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

Again, we have indicated the periodicity of the coefficients.
For example, for the period of $N = 5$, $a_2 = a_{-3}$, which in our example equals $-1/2j$.
Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n}$$

Ex: (3.11 from text book)

Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

This signal is periodic with period N, and we can expand $x[n]$ directly in terms of complex exponentials by using Euler's formula;

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] + \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}]$$

Collecting terms, we find that

$$x[n] = \underbrace{1}_{a_0} + \underbrace{\left(\frac{3}{2} + \frac{1}{2j}\right)}_{a_1} e^{j(2\pi/N)n} + \underbrace{\left(\frac{3}{2} - \frac{1}{2j}\right)}_{a_{-1}} e^{-j(2\pi/N)n} + \underbrace{\left(\frac{1}{2}e^{j\pi/2}\right)}_{a_2} e^{j2(2\pi/N)n} + \underbrace{\left(\frac{1}{2}e^{-j\pi/2}\right)}_{a_{-2}} e^{-j2(2\pi/N)n}$$

Ex: (3.11 from text book)

Thus the Fourier series coefficients for this example are

$$a_0 = 1,$$
$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j,$$
$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}j,$$
$$a_{-2} = -\frac{1}{2}j,$$

$a_k = 0$ for other values of k in the interval of summation (that is one period, N) in the synthesis equation.

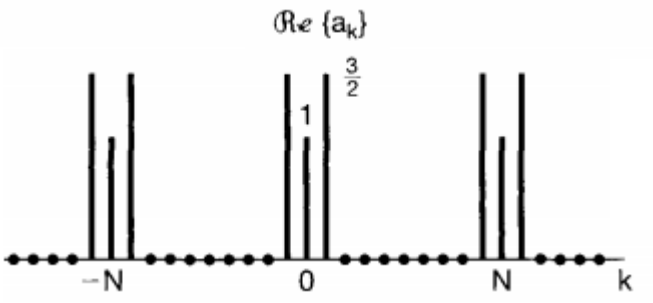
Again, the Fourier coefficients are periodic with period N, so, for example, $a_N = 1$, $a_{3N-1} = \frac{3}{2} + \frac{1}{2}j$, and $a_{2-N} = \frac{1}{2}j$

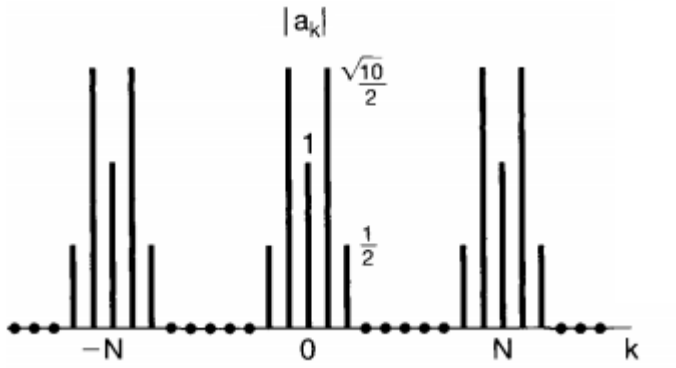
Ch3: Fourier Series

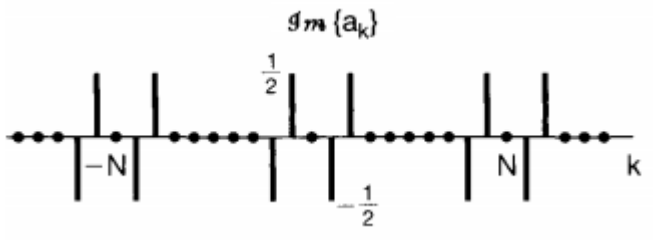
DT Fourier Series

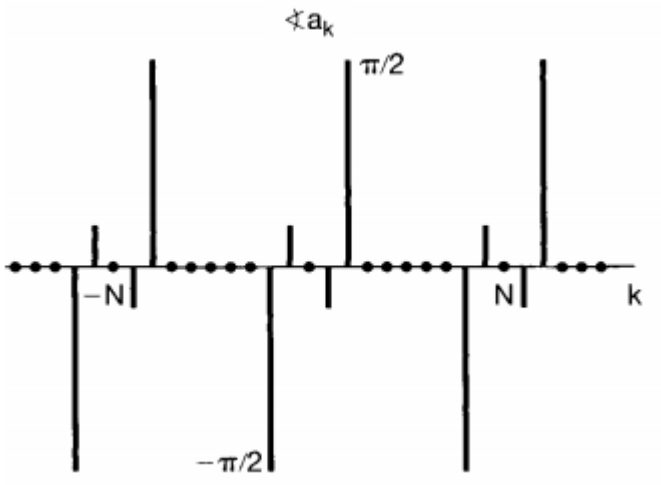
Ex: (3.11 from text book)

The real and imaginary parts, and the magnitude and phase of these coefficients for $N = 10$, are depicted in below figures:









END OF CHAPTER 3

Dr. Yılmaz KALKAN