



AYDIN ADNAN MENDERES UNIVERSITY FACULTY OF ENGINEERING

Department of Electrical and Electronics Engineering

EE303 – Signals and Systems

Prof. Dr. Yılmaz KALKAN



1. Signals & Systems (Chapter 1 of Oppenheim – 6 Hrs.)

Introduction, Continuous-Time and Discrete-Time Signals, Periodicity, Even and Odd Signals, Exponential and Sinusoidal Signals, the Unit Impulse and Unit Step Functions, Continuous-Time and Discrete-Time Systems, Basic System Properties (Memory, Invertibility, Causality, Stability, Time Invariance, Linearity)

2. Linear Time-Invariant (LTI) Systems (Chapter 2 of Oppenheim – 9 Hrs.)

Discrete-Time LTI Systems, Convolution, Continuous-Time LTI Systems, Properties of LTI Systems, LTI Systems with and without Memory, Invertibility of LTI Systems, Causality for LTI Systems, Stability for LTI Systems, Unit Step Response of an LTI System, Causal LTI Systems Described by Differential and Difference Equations, Linear Constant-Coefficient Differential Equations

3. Fourier Series (Chapter 3 of Oppenheim – 6 Hrs.)

Fourier Series Representation of CT Periodic Signals, Convergence of Fourier Series, Gibs Phenomenon, Properties of CT Fourier Series, Fourier Series Representation of DT Periodic Signals, Properties of DT Fourier Series, Fourier Series and LTI Systems

4. CT Fourier Transform (Chapter 4 of Oppenheim – 6 Hrs.)

Op-Amp Terminals & Representation of Aperiodic CT Signals, Convergence of CT Fourier Transforms, Fourier Transform for Periodic Signals, Properties of CT Fourier Transform, Convolution Property, Systems Characterized by Linear Constant-Coefficient Differential Equations

5. DT Fourier Transform (Chapter 5 of Oppenheim – 6 Hrs.)

Representation of Aperiodic DT Signals, Convergence of DT Fourier Transforms, Fourier Transform for Periodic Signals, Properties of DT Fourier Transform, Convolution Property, Systems Characterized by Linear Constant-Coefficient Differential Equations

6. The Laplace Transform (Chapter 9 of Oppenheim – 6 Hrs.)

Introduction, The Region of Convergence for Laplace Transforms, The Inverse Laplace Transform, Pole-Zero Plot, First-Order and Second-Order Systems, Properties of the Laplace Transform, Analysis and Characterization of LTI Systems Using the Laplace Transform, The Unilateral Laplace Transform

7. The Z-Transform (Chapter 10 of Oppenheim – 6 Hrs.)

Introduction, The Region of Convergence for the Z-Transforms, The Inverse Z-Transform, Pole-Zero Plot, First-Order and Second-Order Systems, Properties of the Z-Transform, Analysis and Characterization of LTI Systems Using the Z-Transform, The Unilateral Z-Transform

outline

Ch2: Linear, Time Invariant (LTI) Systems

- In previous chapter we introduced and discussed a number of basic system properties. Two of these, **linearity** and time **invariance**, play a fundamental role in signal and system analysis for two major reasons.

  - First, many physical processes possess these properties and thus can be modeled as linear time-invariant (LTI) systems.

  - LTI systems are amenable to analysis is that any such system possesses the superposition property. As a consequence, if we can represent the input to an LTI system in terms of **a linear combination of a set of basic signals**, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.

Ch2: Linear, Time Invariant (LTI) Systems

Representation of signals in terms of impulses (Reading Assignment)

- One of the important characteristics of the unit impulse, both in discrete time and in continuous time, is that very general signals can be represented as linear combinations of delayed impulses. This fact, together with the properties of superposition and time invariance, will allow us to develop a complete characterization of any LTI system in terms of its response to a unit impulse.
- Such a representation, referred to as the **convolution sum** in the discrete-time case and the **convolution integral** in continuous time, provides considerable analytical convenience in dealing with LTI systems.

- Any discrete-time signal can be represented by using unit impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulses $\delta[n-k]$, where the weights in this linear combination are x[k].

Ch2: Linear, Time Invariant (LTI) Systems

- **DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM (Reading Assignment)**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This is for general case

- How about the system is known as LTI ?
- Consider the response of a linear (but possibly time-varying) system to an arbitrary input x[n].
- We can represent the input through eq. above as a linear combination of shifted unit impulses. Let $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n - k]$.

Ch2: Linear, Time Invariant (LTI) Systems

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM (Reading Assignment)

- Then, from the superposition property for a linear system, the response y[n] of the linear system to the input x[n] in eq. above is simply the weighted linear combination of these basic responses. That is, with the input x[n] to a linear system expressed in the form of eq. above, the output y[n] can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n-k]$$

- Thus, according to eq. above, if we know the response of a linear system to the set of shifted unit impulses, we can construct the response to an arbitrary input.

- If the linear system is also **time invariant**, then these responses to time-shifted unit impulses are all time-shifted versions of each other. Specifically, since $\delta[n-k]$ is a time-shifted version of $\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h[n]$.

Ch2: Linear, Time Invariant (LTI) Systems

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

- That is, h[n] is the output of the LTI system when $\delta[n]$ is the input. Then for an LTI system, previous eq. becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- This result is referred to as the **convolution sum** or superposition sum, and the operation on the right-hand side of eq. above is known as the **convolution** of the sequences x[n] and h[n]. We will represent the operation of convolution symbolically as

$$y[n] = x[n] * h[n]$$

- **LTI systems are completely charecterized by their impulse responses !!**

Ch2: Linear, Time Invariant (LTI) Systems

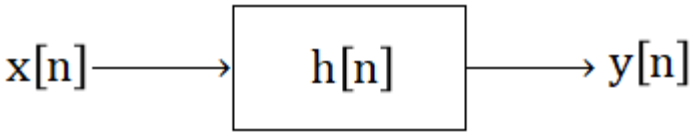
DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

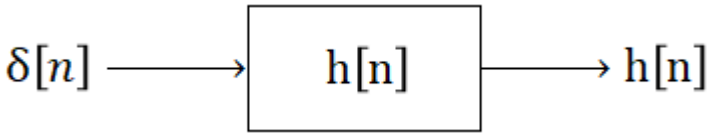
$$y[n] = x[n] * h[n]$$

- ***LTI systems are completely charecterized by their impulse responses !!***

- If you know h[n], you know everthing about that system…



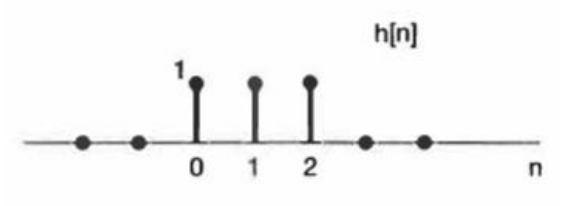
- Why the name of '***IMPULSE RESPONSE***'?

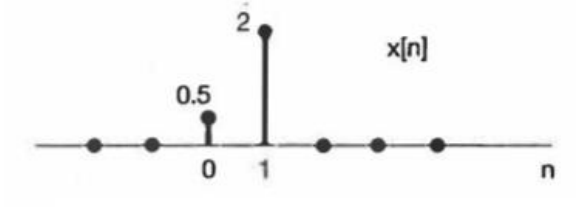


Ch2: Linear, Time Invariant (LTI) Systems

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

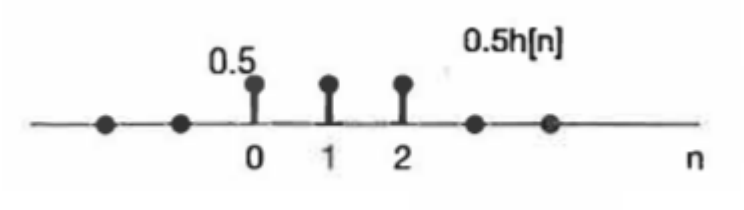
Ex: (2.1 from text book)

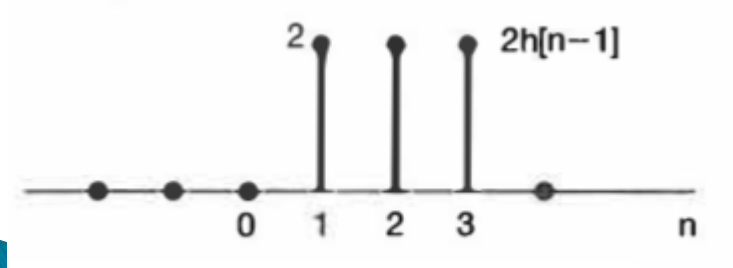


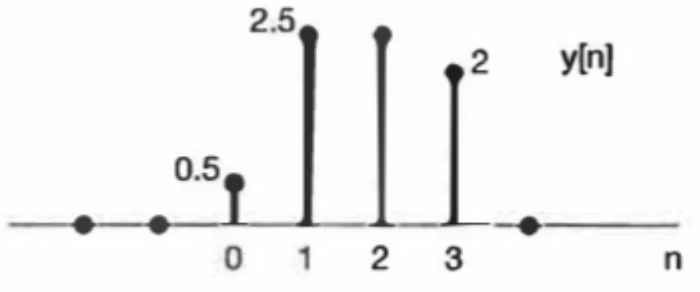




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=0}^{1} x[k]h[n-k] = x[0]h[n] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$







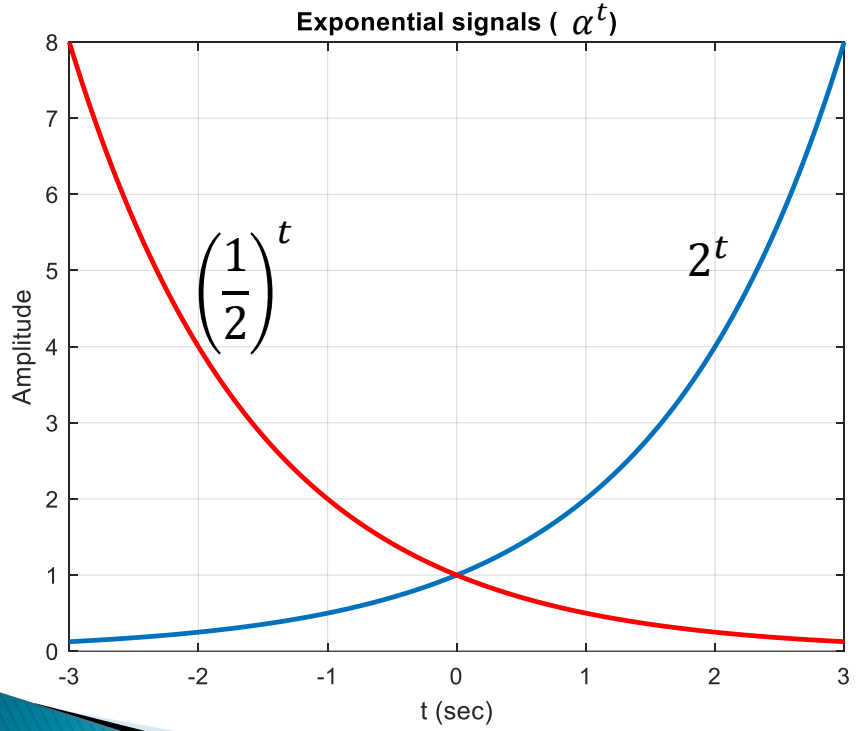
Ch2: Linear, Time Invariant (LTI) Systems

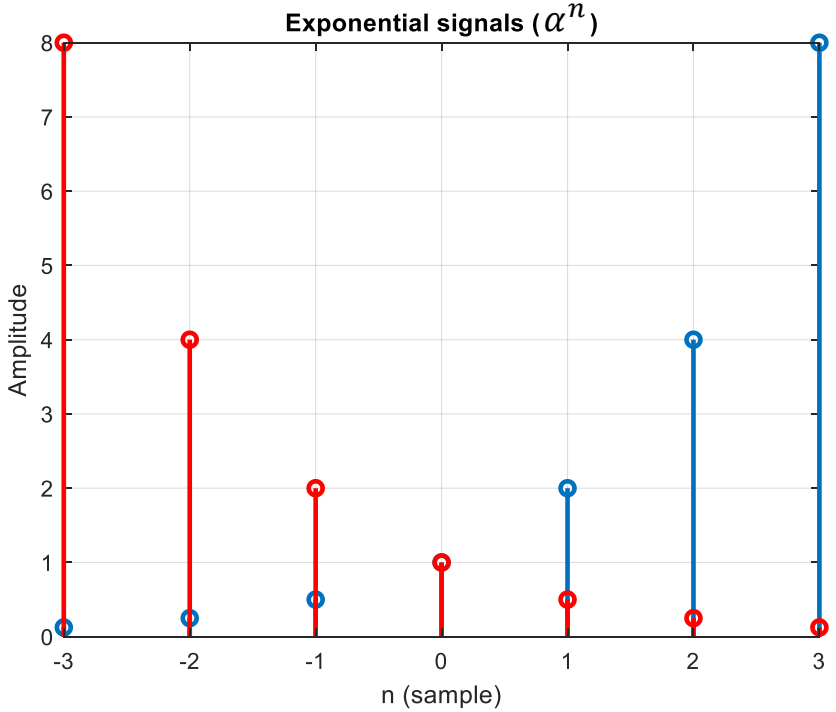
DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

Ex: (2.3 from text book)

Consider an input $x[n]$ and a unit impulse response $h[n]$ given by;

$$x[n] = \alpha^n u[n], \qquad h[n] = u[n] \quad where \;\; 0 < \alpha < 1 \qquad y[n] = ?$$





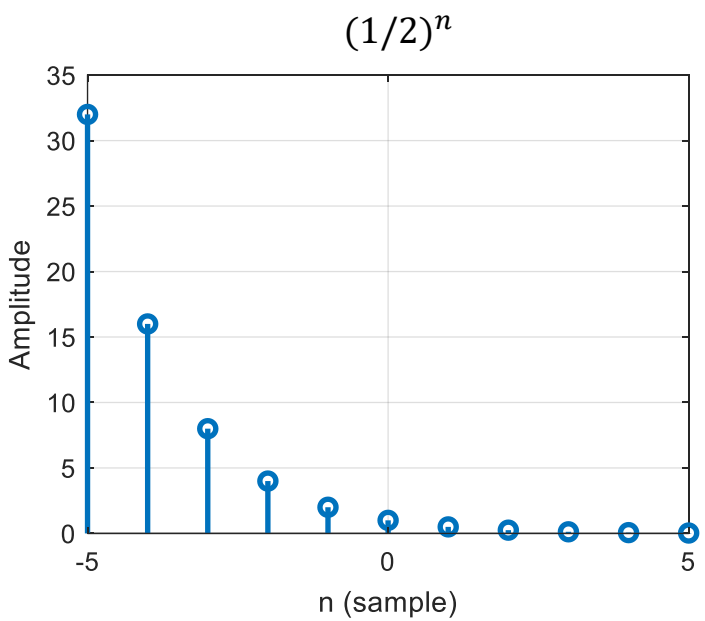
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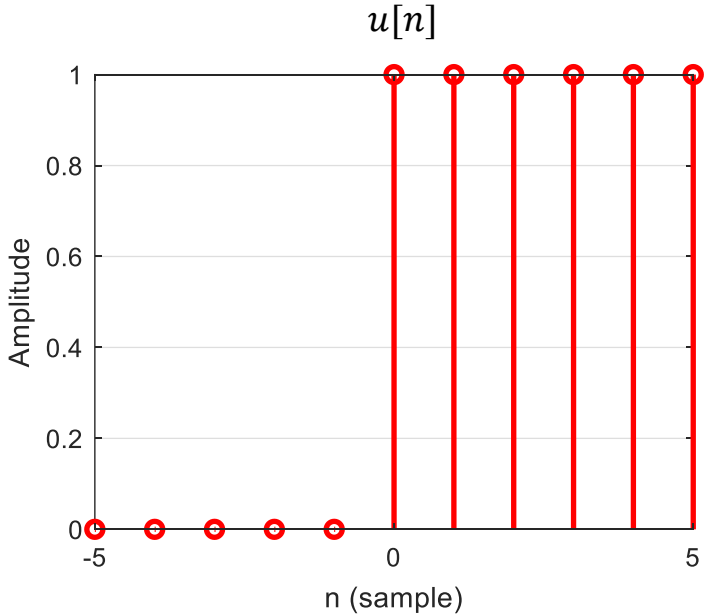
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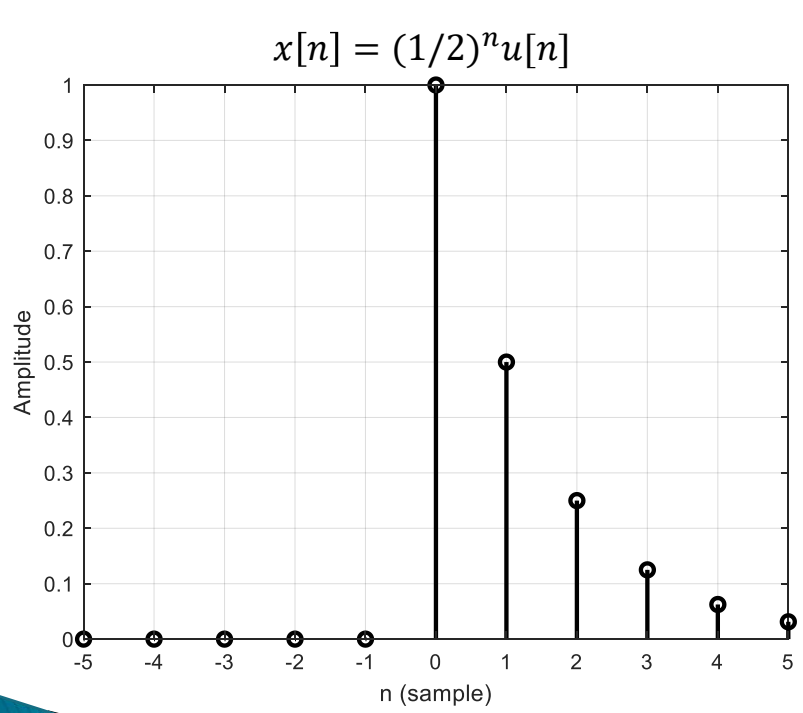
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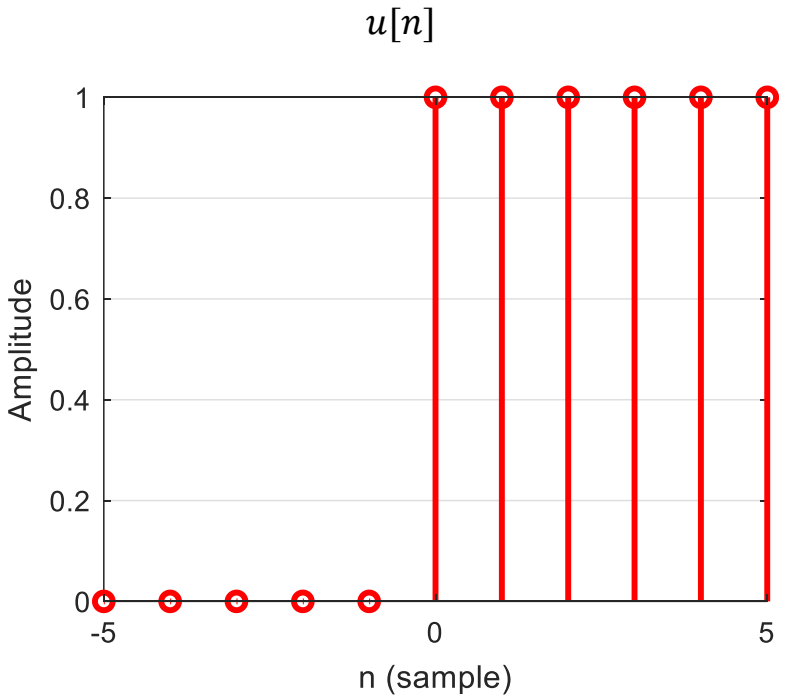
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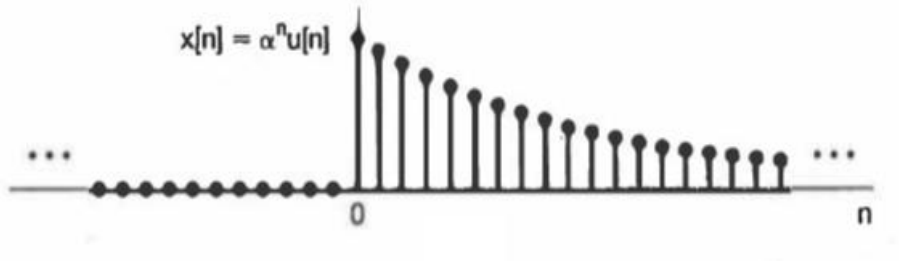
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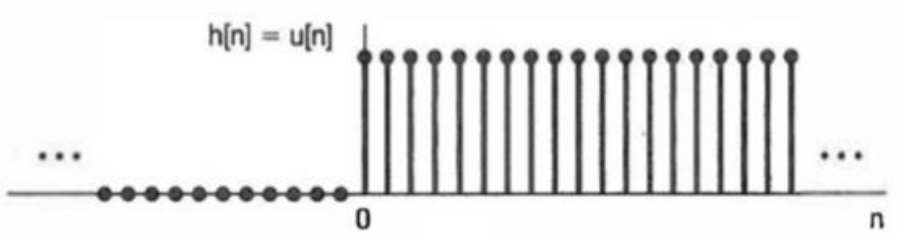
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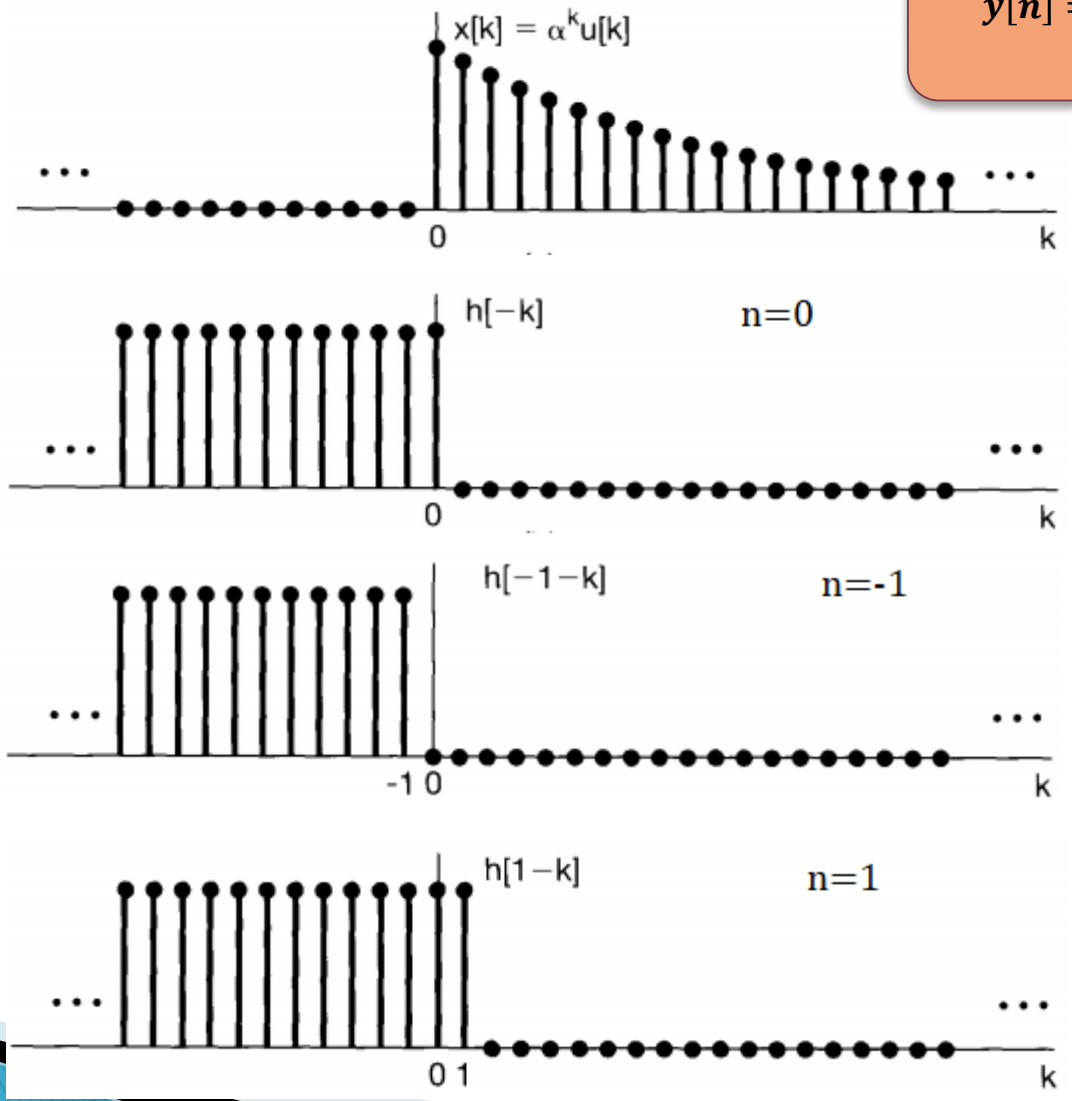
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Ch2: Linear, Time Invariant (LTI) Systems

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

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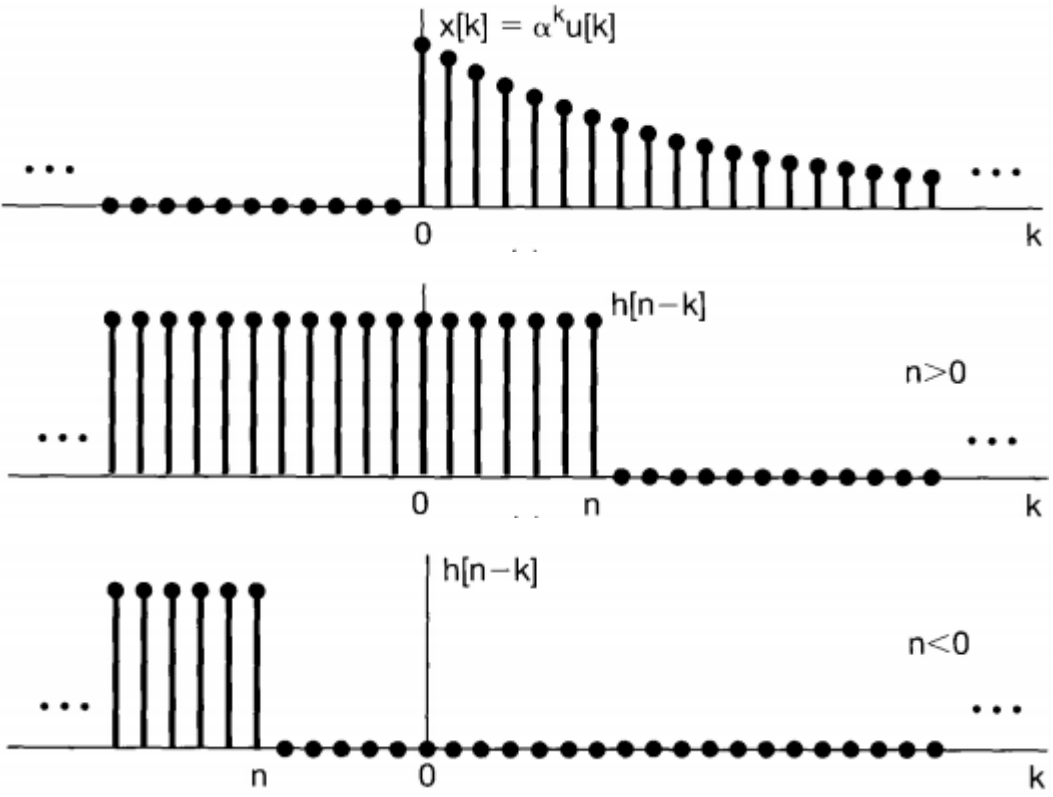
Ch2: Linear, Time Invariant (LTI) Systems

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

Ex: (2.3 from text book)

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Generally;



Thus, for $n < 0$, $x[k]h[n-k] = 0$ for all values of k, hence, we see that $y[n] = 0, n < 0$.

For $n \geq 0$,

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$Thus\ for\ n \geq 0 \quad y[n] = \sum_{k=0}^{n} \alpha^k$$

and using the result of *finite sum formula;*

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad for\ n \geq 0$$

$$\sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}, \alpha \neq 1$$

(finite sum formula)

Ch2: Linear, Time Invariant (LTI) Systems

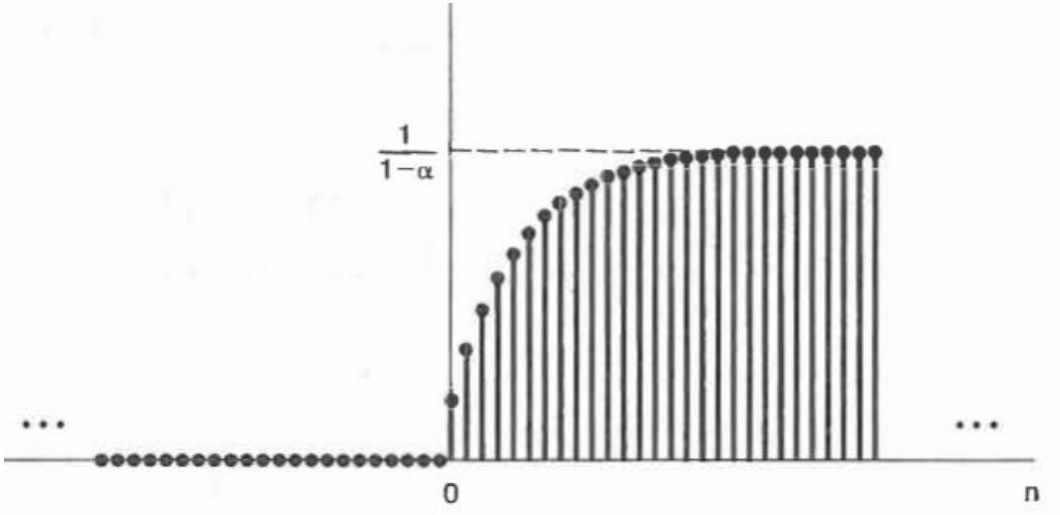
DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

Ex: (2.3 from text book)

Thus for all n,

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad for \;\; n \geq 0$$

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$



Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

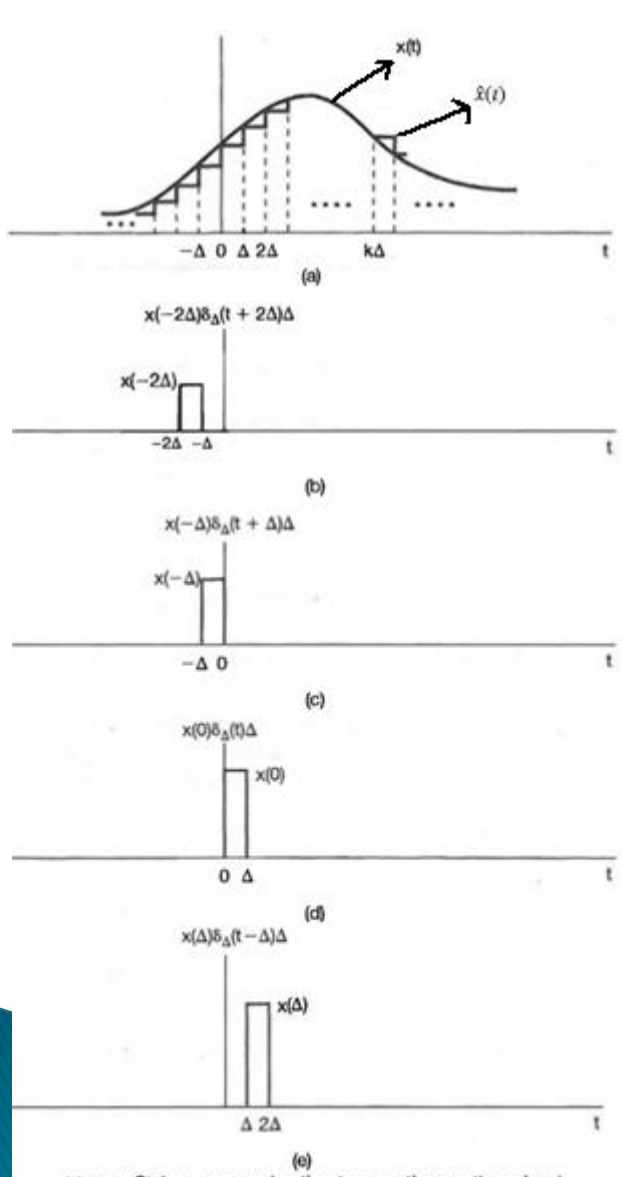


Figure Staircase approximation to a continuous-time signal.

If we define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases},$$

then, since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals $x(t)$. Therefore,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

Also, as $\Delta \to 0$, the summation in above eq. approaches an integral

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau.$$

By using this integral and linearity and time invariancy properties, we can reach the convolution integral for continuous time systems:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \qquad y(t) = x(t) * h(t)$$

Reading Assignment: Sections 2.2.1 and 2.2.2 in Oppenheim.

Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.6 from text book)

Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$x(t) = e^{-at}u(t), \quad a > 0 \quad$ and $\quad h(t) = u(t). \quad y(t) = ?$

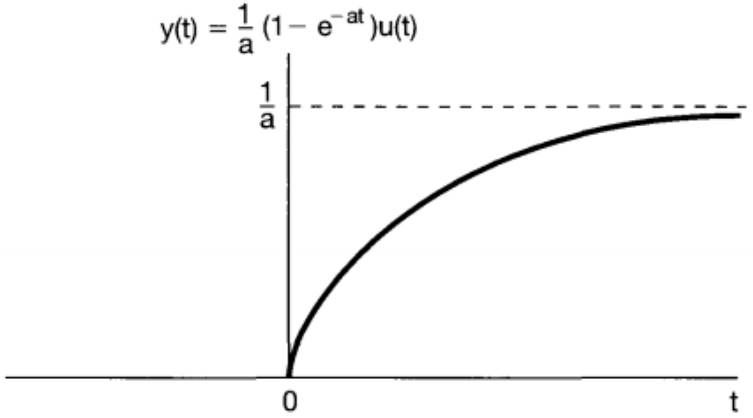
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

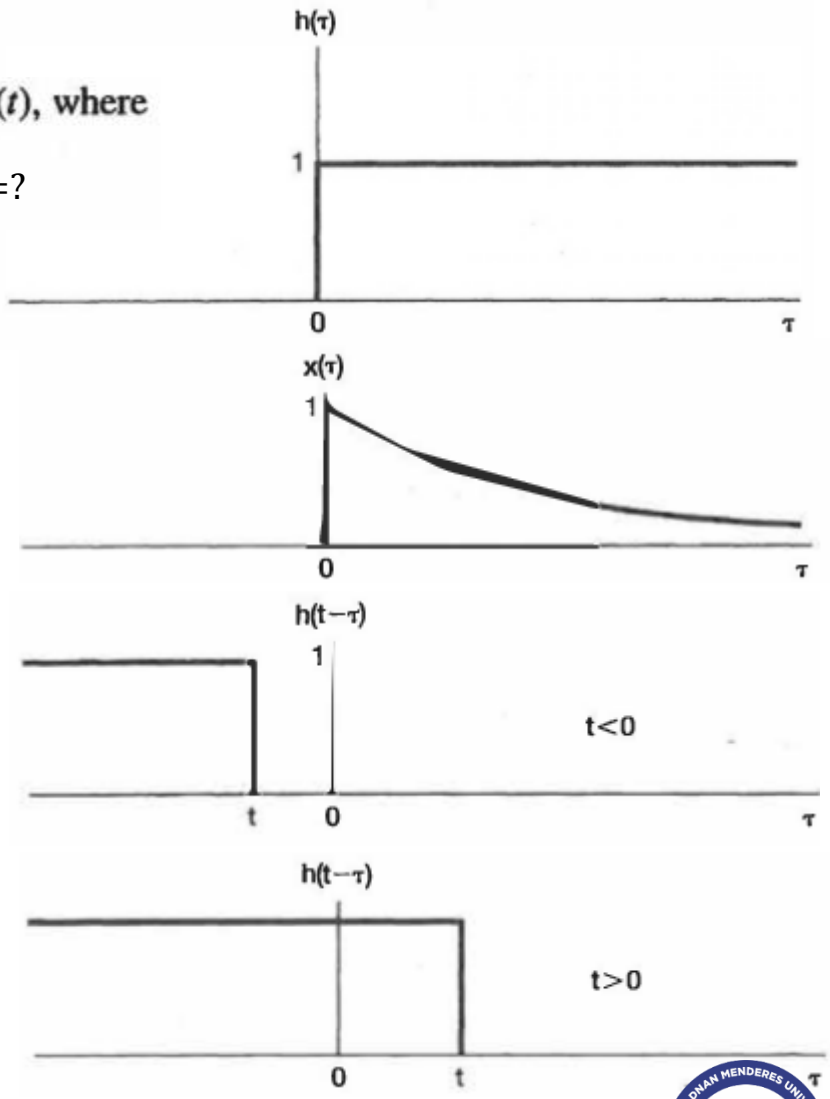
From this expression, we can compute $y(t)$ for $t > 0$:

$$y(t) = \int_0^t e^{-a\tau}d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t = \frac{1}{a}(1 - e^{-at}).$$

Thus, for all t, $y(t)$ is

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t),$$

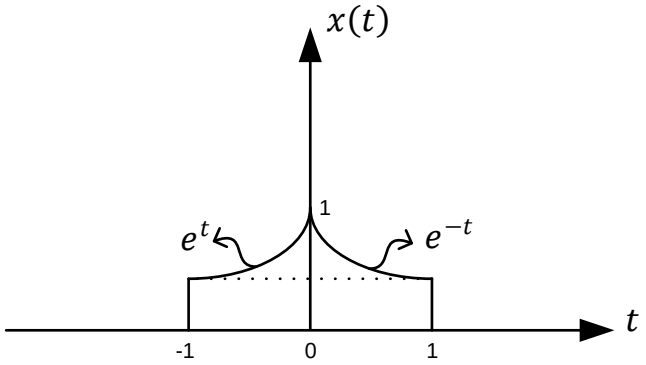


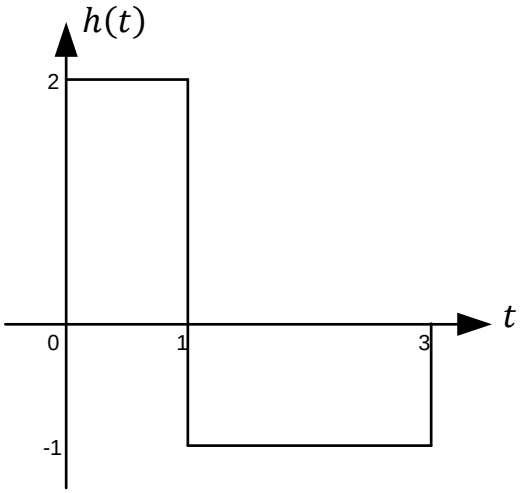


Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex:





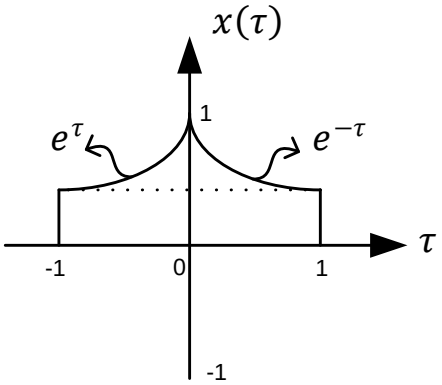
$$y(t) = x(t) * h(t) = ?$$

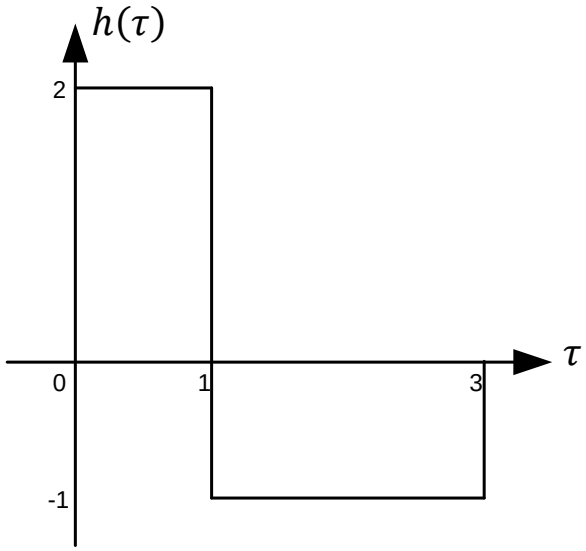
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

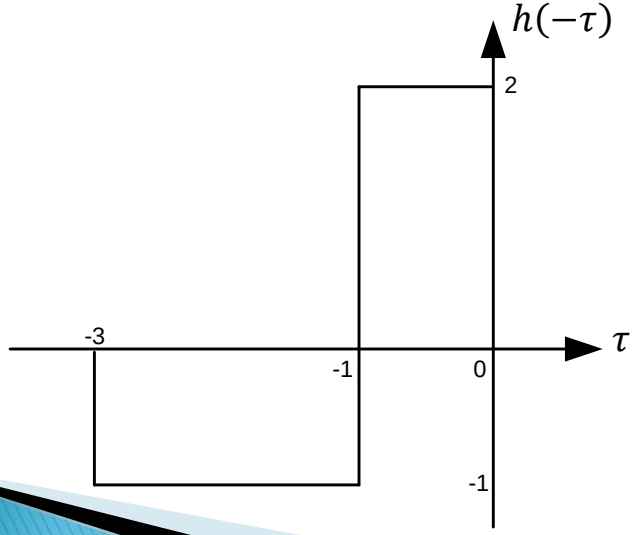
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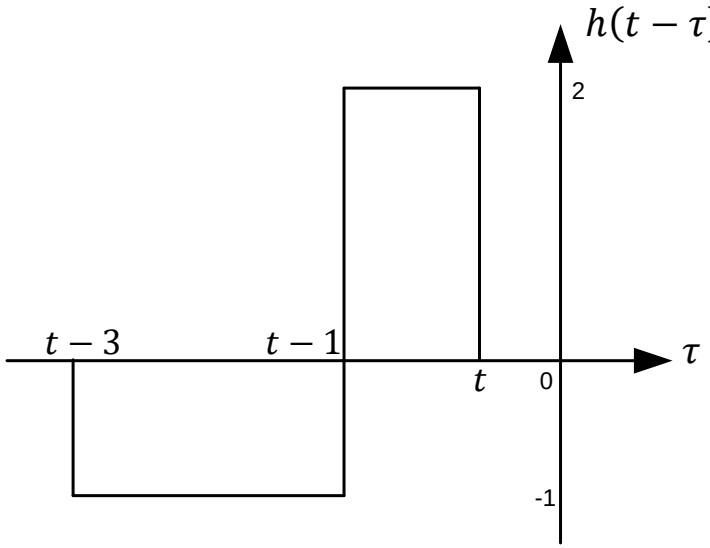
CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex:





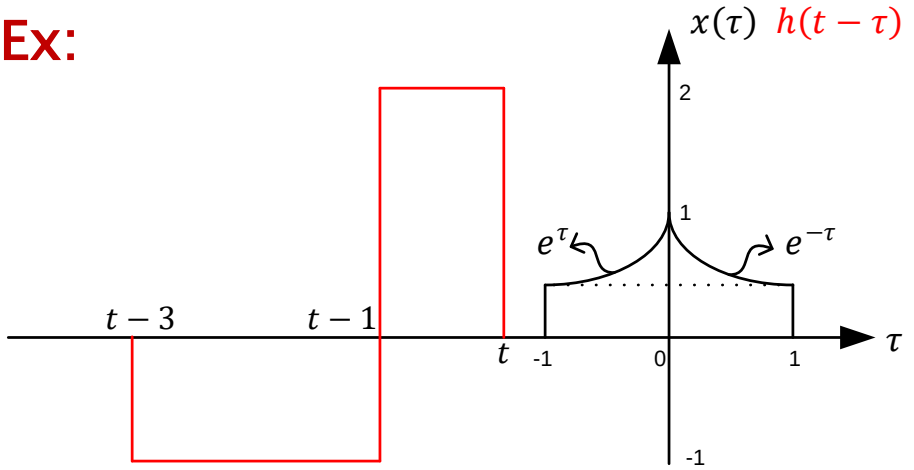




Ch2: Linear, Time Invariant (LTI) Systems

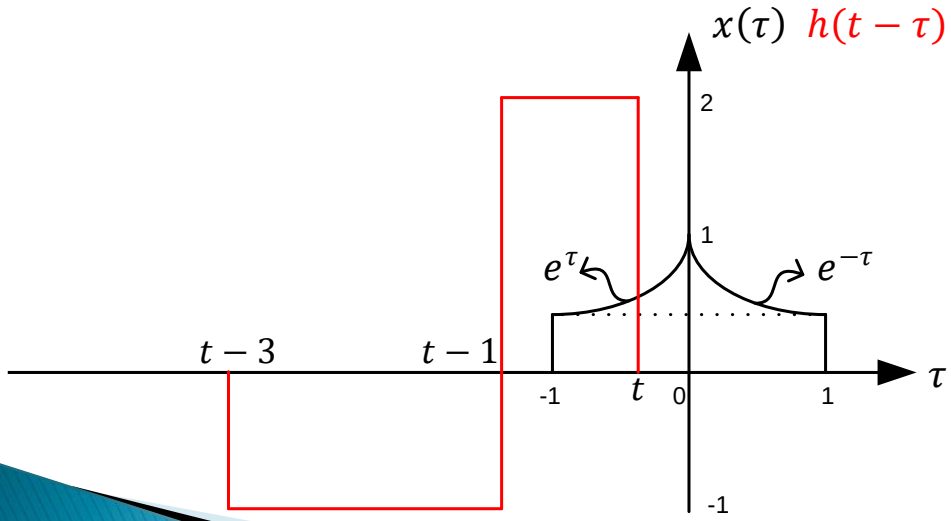
CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex:



For $t \leq -1$ there is no overlap

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$



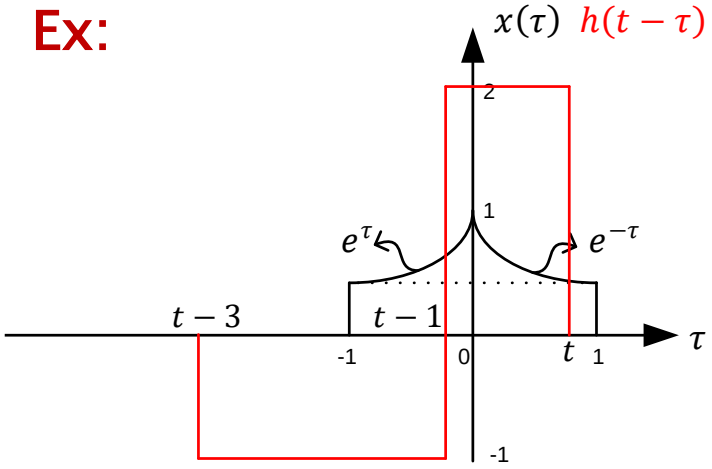
For $-1 \leq t \leq 0$

$$y(t) = \int_{-1}^{t} 2e^{\tau}d\tau = 2\,e^{\tau}\Big|_{-1}^{t} = 2(e^{t} - e^{-1})$$

Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex:



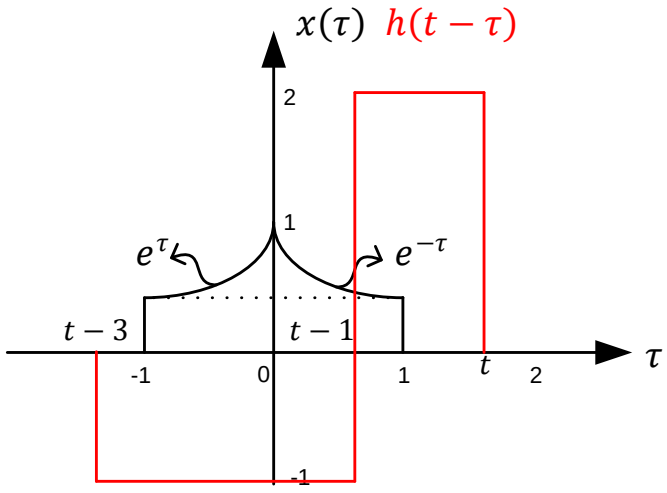
For $0 \le t \le 1$

$$y(t) = \int_{-1}^{t-1} (-1).e^{\tau} d\tau + \int_{t-1}^{0} 2.e^{\tau} d\tau + \int_{0}^{t} 2.e^{-\tau} d\tau$$

$$= -e^{\tau}\Big|_{-1}^{t-1} + 2e^{\tau}\Big|_{t-1}^{0} - 2e^{-\tau}\Big|_{0}^{t}$$

$$= -e^{t-1} + e^{-1} + 2 - 2e^{t-1} - 2e^{-t} + 2$$

$$= -3e^{t-1} + e^{-1} + 4 - 2e^{-t}$$



For $1 \le t \le 2$

$$y(t) = \int_{-1}^{0} (-1).e^{\tau} d\tau + \int_{0}^{t-1} (-1).e^{-\tau} d\tau + \int_{t-1}^{1} 2.e^{-\tau} d\tau$$

$$= -e^{\tau}\Big|_{-1}^{0} + e^{-\tau}\Big|_{0}^{t-1} - 2e^{-\tau}\Big|_{t-1}^{1}$$

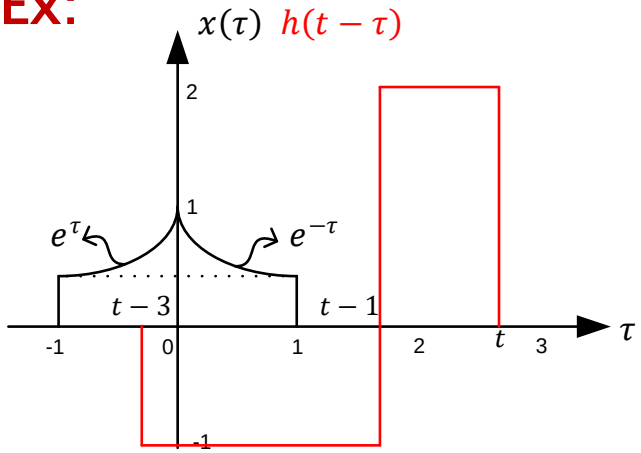
$$= -1 + e^{-1} + e^{1-t} - 1 - 2e^{-1} + 2e^{1-t}$$

$$= -2 - e^{-1} + 3e^{1-t}$$

Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex:



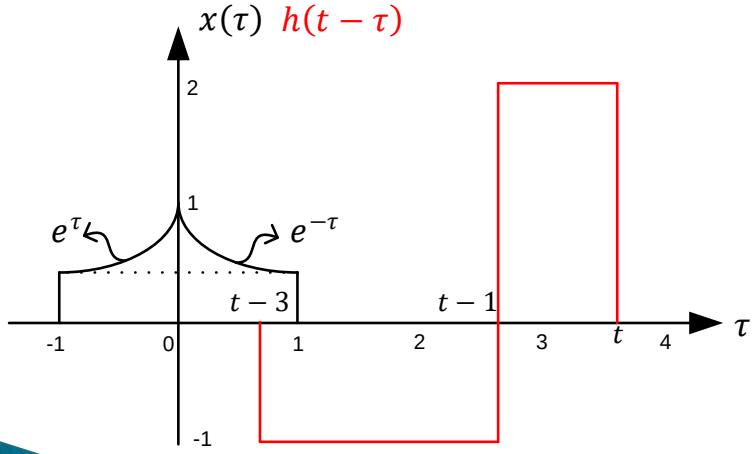
For $2 \le t \le 3$

$$y(t) = \int_{t-3}^{0} (-1).e^{\tau} d\tau + \int_{0}^{1} (-1).e^{-\tau} d\tau$$

$$= -e^{\tau}\Big|_{t-3}^{0} + e^{-\tau}\Big|_{0}^{1}$$

$$= -1 + e^{t-3} + e^{-1} - 1$$

$$= -2 + e^{t-3} + e^{-1}$$



For $3 \le t \le 4$

$$y(t) = \int_{t-3}^{1} (-1).e^{-\tau} d\tau$$

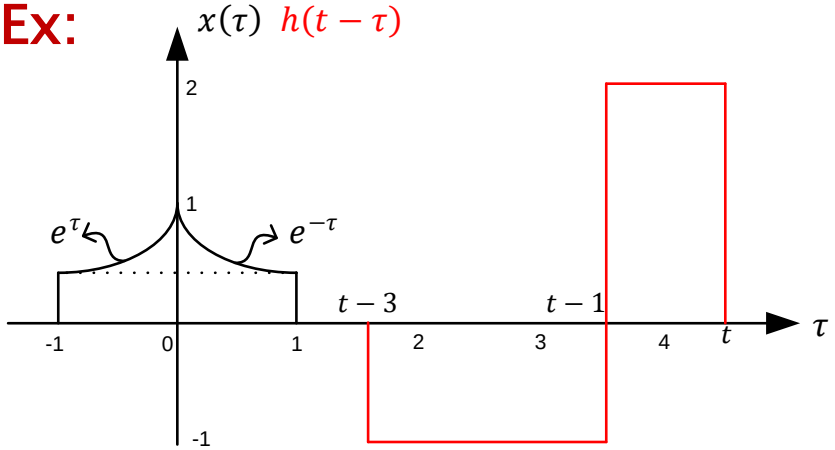
$$= e^{-\tau}\Big|_{t-3}^{1}$$

$$= e^{-1} - e^{3-t}$$

Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex:



For $t \geq 4$ there is no overlap

$$y(t) = 0$$

Finally;

$$y(t) = \begin{cases} 0 & , \quad t \leq -1 \\ 2(e^{t} - e^{-1}) & , \quad -1 \leq t \leq 0 \\ -3e^{t-1} + e^{-1} + 4 - 2e^{-t} & , \quad 0 \leq t \leq 1 \\ -2 - e^{-1} + 3e^{1-t} & , \quad 1 \leq t \leq 2 \\ -2 + e^{t-3} + e^{-1} & , \quad 2 \leq t \leq 3 \\ e^{-1} - e^{3-t} & , \quad 3 \leq t \leq 4 \\ 0 & , \quad t \geq 4 \end{cases}$$

Ch2: Linear, Time Invariant (LTI) Systems

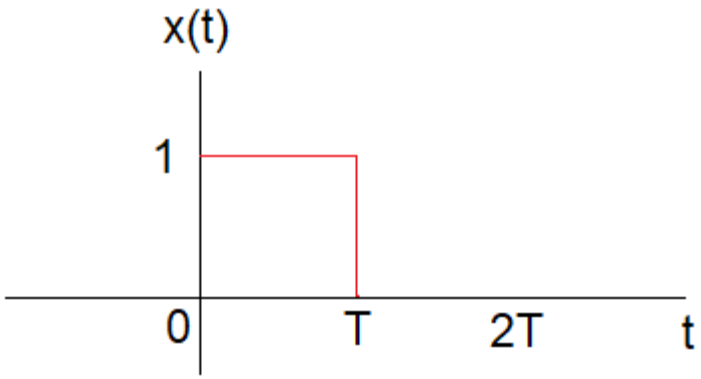
CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

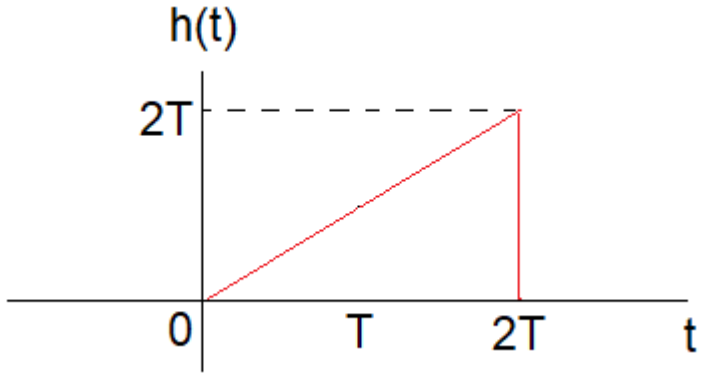
Ex: (2.7 from text book)

Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}.$$

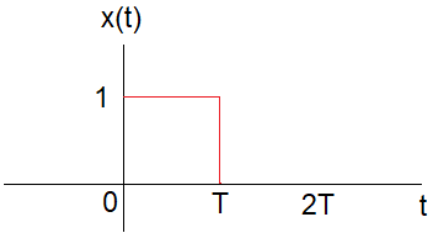


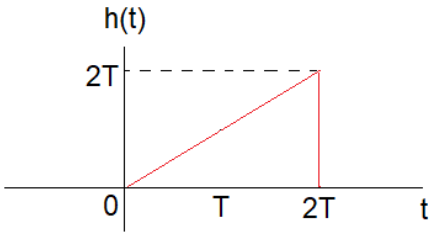


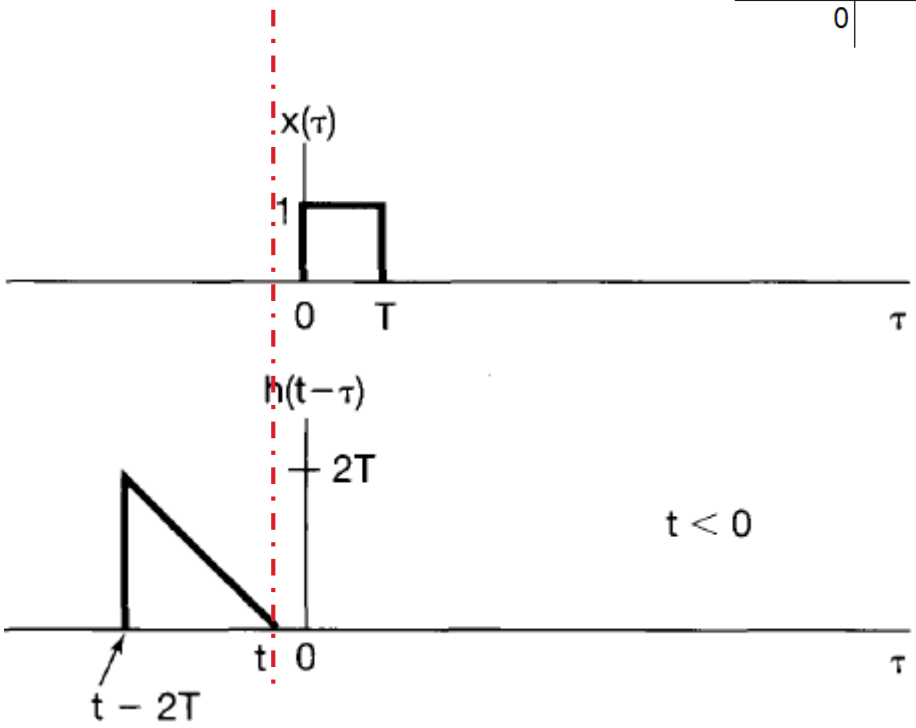
Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.7 from text book)







There is no overlap

$$y(t) = x(t) * h(t)$$

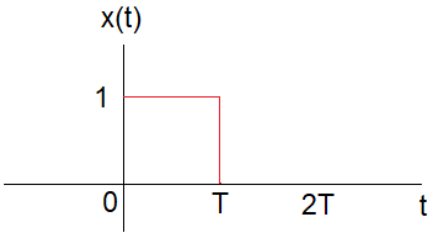
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

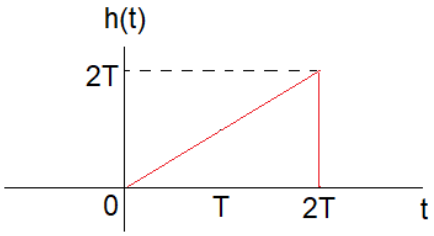
$$= 0$$

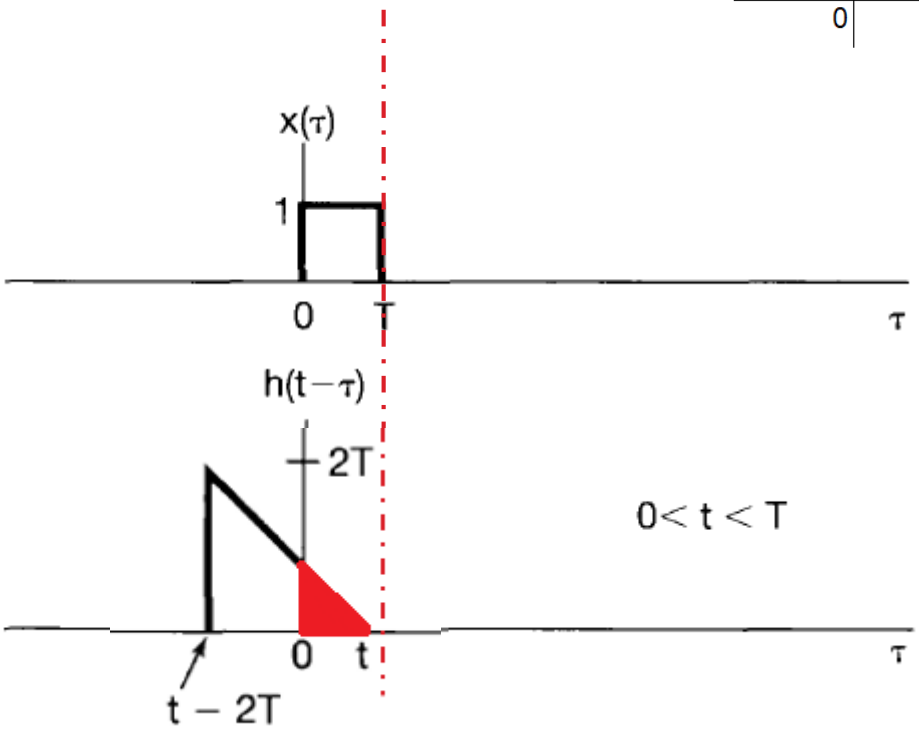
Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.7 from text book)







$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

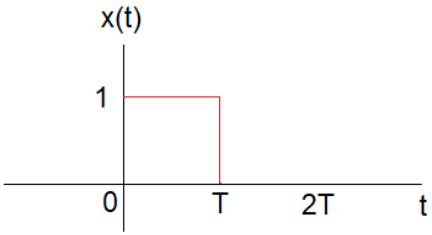
$$= \int_{0}^{t} 1.(t-\tau)d\tau$$

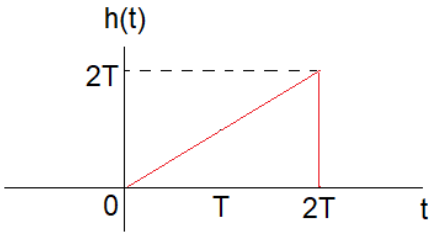
$$= \frac{1}{2}t^2$$

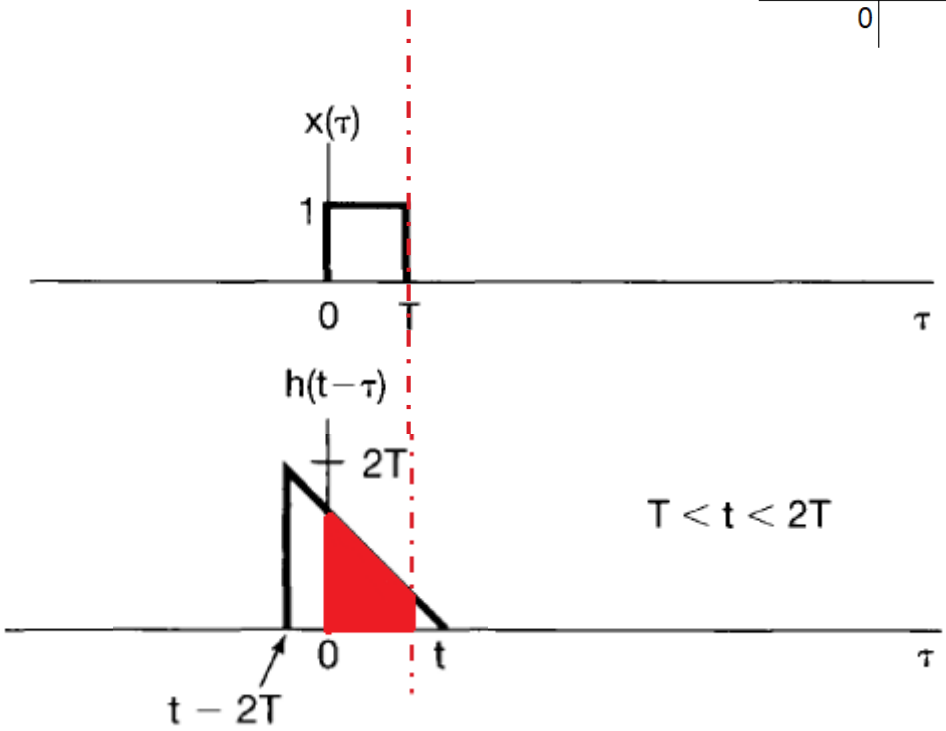
Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.7 from text book)







$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{T} x(\tau)h(t-\tau)d\tau$$

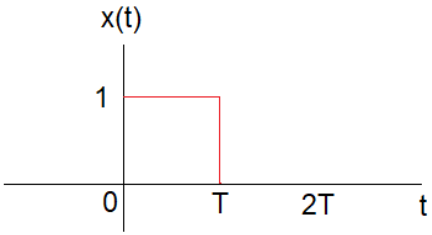
$$= \int_{0}^{T} 1.(t-\tau)d\tau$$

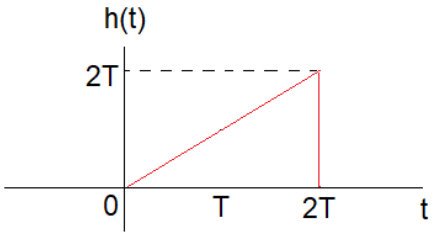
$$= Tt - \frac{1}{2}T^2$$

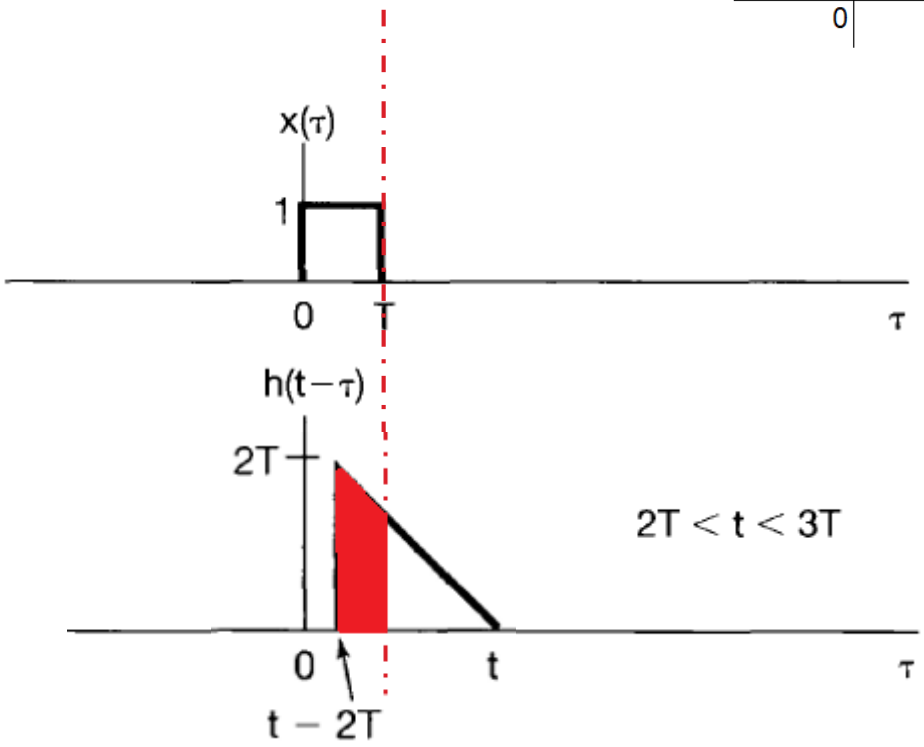
Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.7 from text book)







$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-2T}^{T} x(\tau)h(t-\tau)d\tau$$

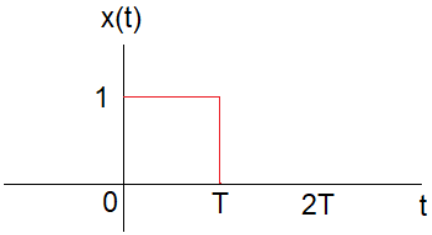
$$= \int_{t-2T}^{T} 1.(t-\tau)d\tau$$

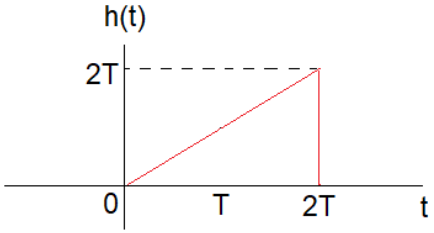
$$= Tt - \frac{1}{2}t^2 + \frac{3}{2}T^2$$

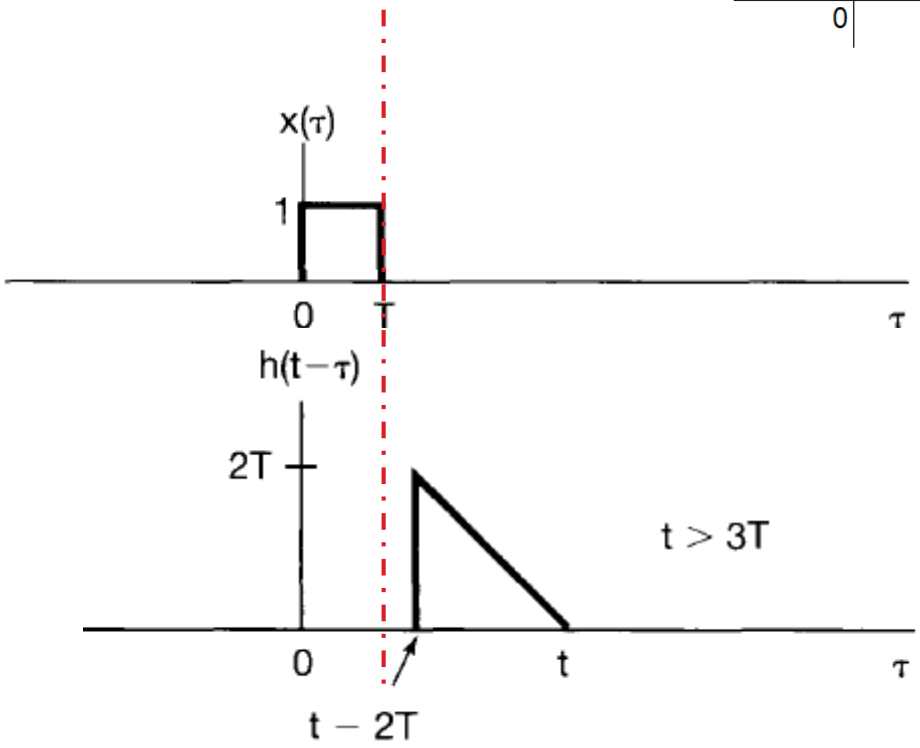
Ch2: Linear, Time Invariant (LTI) Systems

CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.7 from text book)







There is no overlap

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{2T}^{\infty} x(\tau)h(t-\tau)d\tau$$

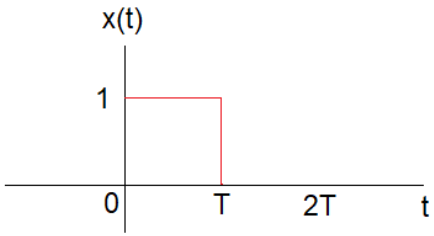
$$= 0$$

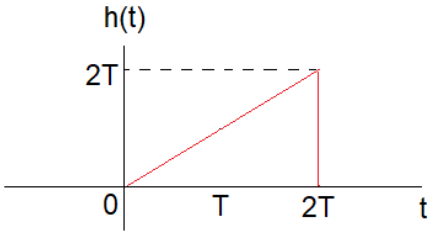
Ch2: Linear, Time Invariant (LTI) Systems

Consider the convolution of the following two signals:

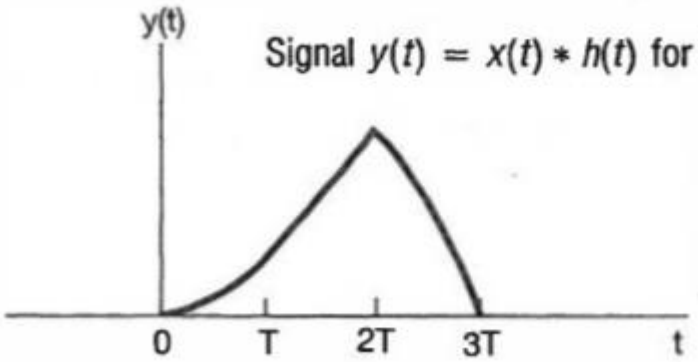
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}.$$





$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases},$$



Signal $y(t) = x(t) * h(t)$ for Example 2.7.

Ch2: Linear, Time Invariant (LTI) Systems

DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

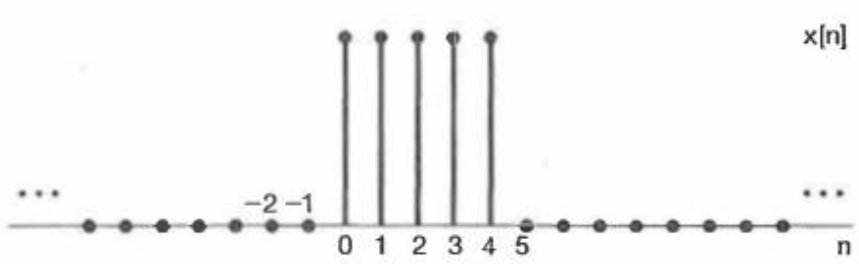
Ex: (2.4 from text book)

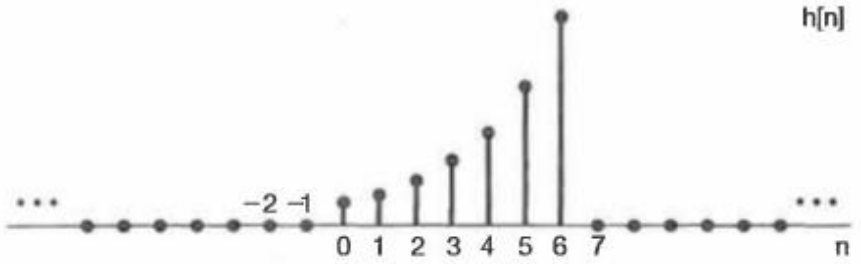
As a further example, consider the two sequences

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

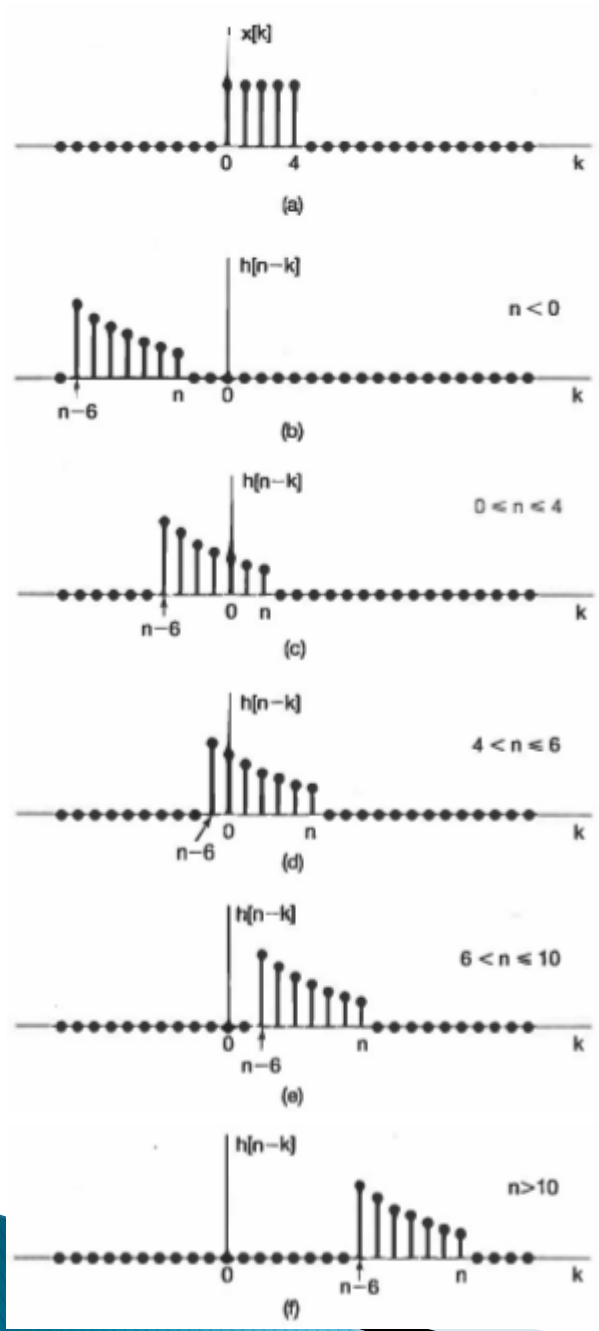
and

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}.$$





$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



Interval 1. For $n < 0$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and consequently, $y[n] = 0$.

Interval 2. For $0 \le n \le 4$,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

Thus, in this interval,

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}. \tag{2.14}$$

We can evaluate this sum using the finite sum formula, eq. (2.13). Specifically, changing the variable of summation in eq. (2.14) from k to $r = n - k$, we obtain

$$y[n] = \sum_{r=0}^{n} \alpha^{r} = \frac{1-\alpha^{n+1}}{1-\alpha}.$$

Interval 3. For $n > 4$ but $n - 6 \le 0$ (i.e., $4 < n \le 6$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Thus, in this interval,

$$y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \alpha^{n} \sum_{k=0}^{4} (\alpha^{-1})^{k} = \alpha^{n} \frac{1-(\alpha^{-1})^{5}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

Interval 4. For $n > 6$ but $n - 6 \le 4$ (i.e., for $6 < n \le 10$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases} \qquad y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}$$

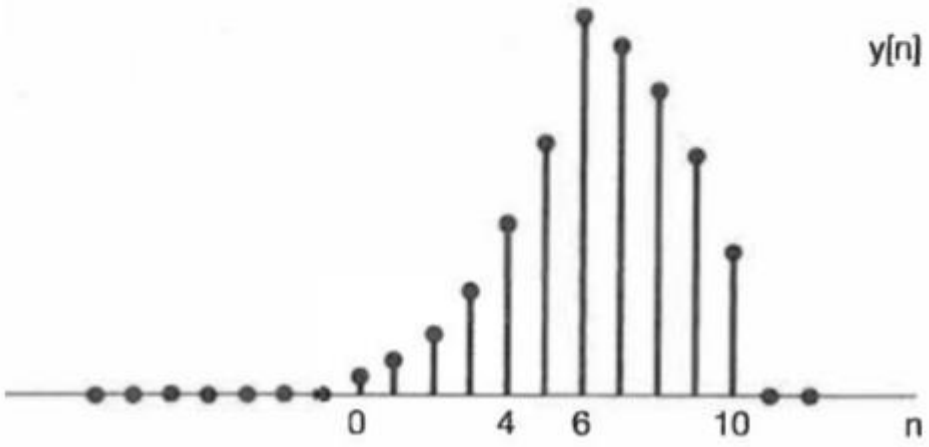
Letting $r = k - n + 6$, we obtain

$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^{6} \sum_{r=0}^{10-n} (\alpha^{-1})^{r} = \alpha^{6} \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

Interval 5. For $n - 6 > 4$, or equivalently, $n > 10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and hence,

$$y[n] = 0.$$

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases},$$



Ch2: Linear, Time Invariant (LTI) Systems

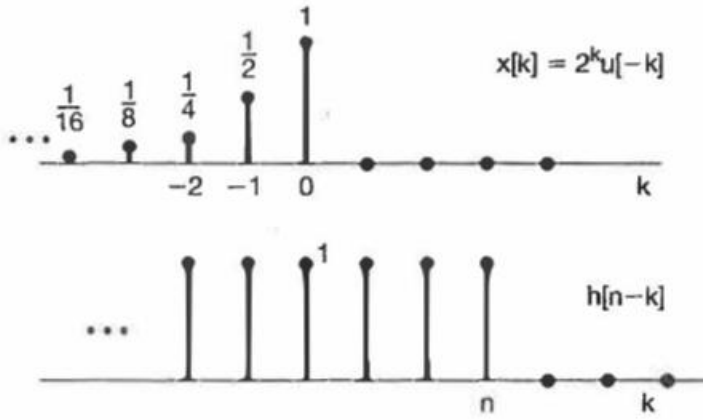
DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

Ex: (2.5 from text book)

Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows:

$$x[n] = 2^n u[-n],$$
$$h[n] = u[n].$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



$x[k]$ is zero for $k > 0$ and $h[n-k]$ is zero for $k > n$. We also observe that, regardless of the value of n, the sequence $x[k]h[n-k]$ always has nonzero samples along the k-axis. When $n \geq 0$, $x[k]h[n-k]$ has nonzero samples in the interval $k \leq 0$. It follows that, for $n \geq 0$,

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k. \tag{2.19}$$

To evaluate the infinite sum in eq. (2.19), we may use the *infinite sum formula*,

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \quad 0 < |\alpha| < 1. \tag{2.20}$$

Changing the variable of summation in eq. (2.19) from k to $r = -k$, we obtain

$$\sum_{k=-\infty}^{0} 2^k = \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-(1/2)} = 2. \tag{2.21}$$

Thus, $y[n]$ takes on a constant value of 2 for $n \geq 0$.

Ch2: Linear, Time Invariant (LTI) Systems

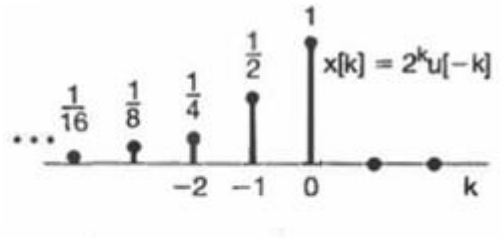
DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

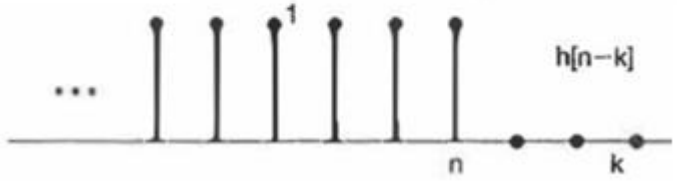
Ex: (2.5 from text book)

Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows:

$$x[n] = 2^n u[-n],$$
$$h[n] = u[n].$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



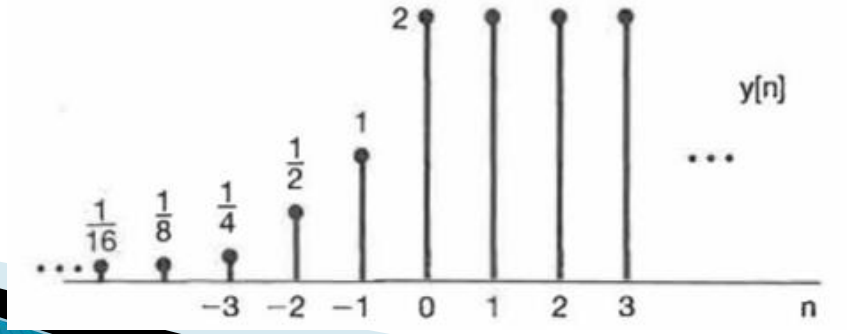


When $n < 0$, $x[k]h[n-k]$ has nonzero samples for $k \leq n$. It follows that, for $n < 0$,

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{n} 2^k. \qquad (2.22)$$

By performing a change of variable $l = -k$ and then $m = l + n$, we can again make use of the infinite sum formula, eq. (2.20), to evaluate the sum in eq. (2.22). The result is the following for $n < 0$:

$$y[n] = \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n} = \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}. \qquad (2.23)$$



Ch2: Linear, Time Invariant (LTI) Systems

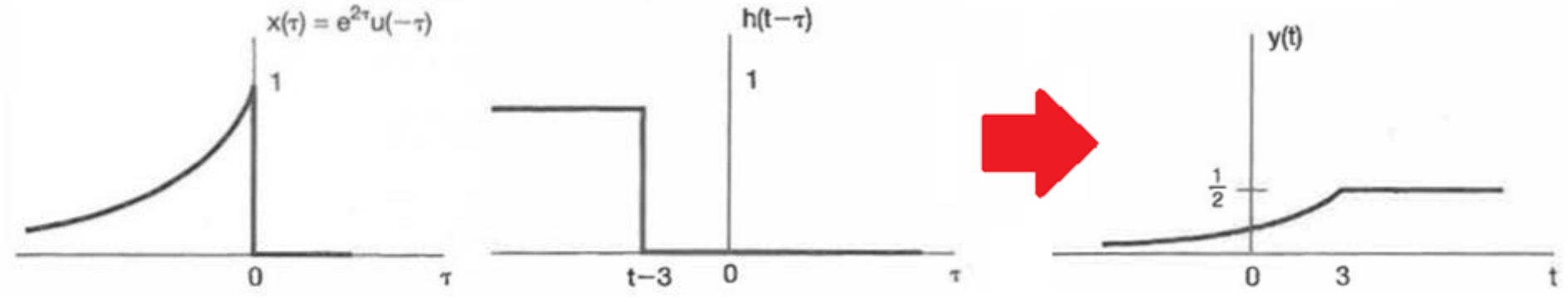
CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

Ex: (2.8 from text book)

Let $y(t)$ denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t),$$
$$h(t) = u(t-3).$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



When $t-3 \le 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < t-3$,

and the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\,d\tau = \frac{1}{2}e^{2(t-3)}$$

For $t-3 \ge 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < 0$,

and the convolution integral now becomes

$$y(t) = \int_{-\infty}^{0} e^{2\tau}\,d\tau = \frac{1}{2}$$

Ch2: Linear, Time Invariant (LTI) Systems

PROPERTIES OF LTI SYSTEMS

- Up to here, we developed the extremely important representations of continuous-time and discrete-time **LTI** systems in terms of their unit impulse responses.

- In discrete time the representation takes the form of the **convolution sum**, while its continuous-time counterpart is the **convolution integral**, both of which we repeat here for convenience:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- **LTI systems have a number of properties not possessed by other systems**, beginning with the very special representations that they have in terms of convolution sums and integrals. In the remainder of this section, we explore some of the most basic and important of these properties.

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

1) The Commutative Property

- A basic property of convolution in both continuous and discrete time is that it is a ***commutative*** operation.

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k],$$

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

2) The Distributive Property

- Another basic property of convolution is the ***distributive*** property. Specifically, convolution distributes over addition.

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n],$$

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$

3) The Associative Property

- Another important and useful property of convolution is that it is ***associative***.

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n],$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t).$$

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

4) Memory (LTI Systems with and without Memory)

For a memoryless LTI system, impulse response has the form;

$$h[n] = K\delta[n], \qquad \text{in DT}$$

or equivaletly;

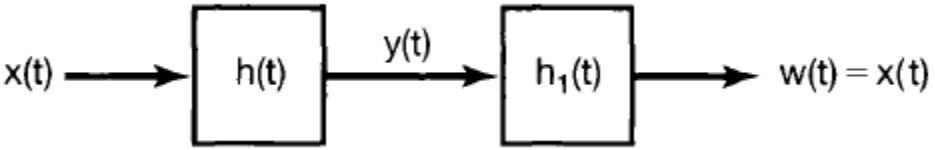
$$h(t) = K\delta(t). \qquad \text{İn CT}$$

otherwise, system has MEMORY.

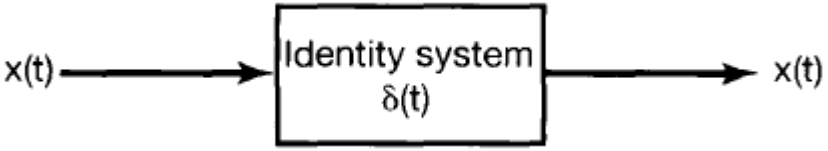
Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

5) Invertibility of LTI Sytems

For a invertible LTI system,



or equivaletly;



For an LTI system, system is invertible iff we can find a $h_I[n]$ or $h_I(t)$ as;

$$h[n] * h_1[n] = \delta[n] \qquad \text{in DT}$$

or

$$h(t) * h_1(t) = \delta(t) \qquad \text{in CT}$$

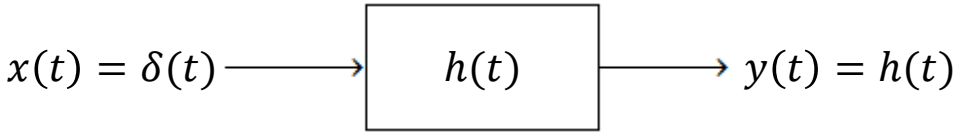
Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

5) Invertibility of LTI Sytems

Example:

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0).$$



$$h(t) = \delta(t - t_0).$$

$$y(t) = x(t) * h(t)$$

Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0).$$

$$h(t) * h_1(t) = \delta(t)$$

That is, if we take

$$h_1(t) = \delta(t + t_0),$$

then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

Remember: $f(t) * \delta(t - t_0) = f(t - t_0)$
If $f(t) = \delta(t + t_0)$ then $f(t - t_0) = \delta(t - t_0 + t_0) = \delta(t)$

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

5) Invertibility of LTI Sytems

Example:

Consider an LTI system with impulse response

$$h[n] = u[n].$$

Using the convolution sum, we can calculate the response of this system to an arbitrary input:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]u[n-k]. \quad \Rightarrow \quad y[n] = \sum_{k=-\infty}^{n} x[k].$$

Such a system is invertible and its inverse is; $\Rightarrow$ $y[n] = x[n] - x[n-1],$

And impulse response of the inverse system is; $\Rightarrow$ $h_1[n] = \delta[n] - \delta[n-1].$

Verify it. $\Rightarrow$

$$\begin{aligned} h[n] * h_1[n] &= u[n] * \{\delta[n] - \delta[n-1]\} \\ &= u[n] * \delta[n] - u[n] * \delta[n-1] \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned}$$

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

6) Causality of LTI Sytems

For a causal LTI system,

$h[n] = 0 \quad \text{for } n < 0.$ in DT & $h(t) = 0 \quad \text{for } t < 0,$ in CT

7) Stability of LTI Sytems (BIBO Stability)

For a stable LTI system,

$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty,$ in DT & $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty.$ in CT

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

8) Unit Step Response of an LTI Sytem

- We have seen that the representation of an LTI system in terms of its unit impulse response allows us to obtain very explicit characterizations of system properties.

- Specifically, since *h[n]* or *h(t)* completely determines the behavior of an LTI system, we have been able to relate system properties such as stability and causality to properties of the impulse response.

- There is another signal that is also used quite often in describing the behavior of LTI systems: **the *unit step response, s[n]* or *s(t),*** corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$.

$$s[n] = u[n] * h[n]. \Rightarrow s[n] = \sum_{k=-\infty}^{n} h[k]. \Rightarrow h[n] = s[n] - s[n-1].$$

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \qquad h(t) = \frac{ds(t)}{dt} = s'(t).$$

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

Example:

The input-output difference equation of a system is given below. Using the impulse response of the system, comment on whether this system is memoryless, causal, stable or not. Obtain the impulse response of the reverse system.

$$y[n] = x[n] + \frac{1}{4}x[n-2]$$

First let's obtain the system's impulse response $h[n]$ for $x[n] = \delta[n]$

$$h[n] = \delta[n] + \frac{1}{4}\delta[n-2]$$

Memory: The system is not memoryless because there is $\delta[n-2]$ term in $h[n]$ expression.

Causality: For $n < 0$ $\delta[n] = 0$ and also $\delta[n-2] = 0$. So, for $n < 0$ $h[n] = 0$ and then the system is causal.

Stability: The following condition must be satisfied:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$\sum_{k=-\infty}^{\infty} |\delta[k]| + \frac{1}{4}\sum_{k=-\infty}^{\infty} |\delta[k-2]| = 1 + \frac{1}{4}.1 = \frac{5}{4} < \infty$$

So, the system is stable.

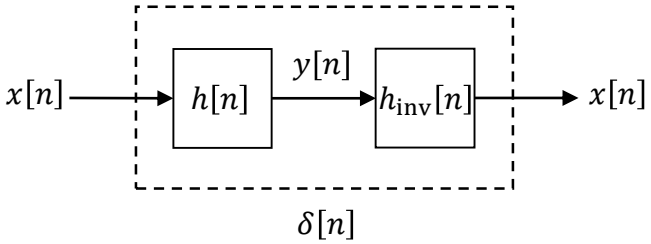
Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

Example:

$$h[n] = \delta[n] + \frac{1}{4}\delta[n-2]$$

Impulse response of the reverse system: It must be,

$$h[n] * h_{\text{inv}}[n] = \delta[n]$$



$$\left(\delta[n] + \frac{1}{4}\delta[n-2]\right) * h_{\text{inv}}[n] = \delta[n]$$

$$(\delta[n] * h_{\text{inv}}[n]) + \frac{1}{4}(\delta[n-2] * h_{\text{inv}}[n]) = \delta[n]$$

$$h_{\text{inv}}[n] + \frac{1}{4}h_{\text{inv}}[n-2] = \delta[n]$$

Let us consider the equation $h[n] * h_{\text{inv}}[n] = \delta[n]$

Since the original system is causal,

the first index of $h[n]$ is $n = 0$.

As it is known, the first index of $\delta[n]$ is also $n = 0$.

In addition, the first index of $\delta[n]$, which is the result of the convolution process, must be the sum of the first index of $h[n]$ and the first index of $h_{\text{inv}}[n]$.

So the first index of $h_{\text{inv}}[n]$ must be $n = 0$. Then $h_{\text{inv}}[n] = 0$ for $n < 0$ and the inverse system is also causal.

Ch2: LTI Systems – PROPERTIES OF LTI SYSTEMS

Example:

So let's try to find $h_{\text{inv}}[n]$ intuitively by giving values to n, starting from $n = 0$:

$$h_{\text{inv}}[n] + \frac{1}{4}h_{\text{inv}}[n-2] = \delta[n]$$

For $n = 0$; $\quad h_{\text{inv}}[0] + \frac{1}{4}h_{\text{inv}}[-2] = \delta[0] \quad \Rightarrow \quad h_{\text{inv}}[0] = 1$

For $n = 1$; $\quad h_{\text{inv}}[1] + \frac{1}{4}h_{\text{inv}}[-1] = \delta[1] \quad \Rightarrow \quad h_{\text{inv}}[1] = 0$

For $n = 2$; $\quad h_{\text{inv}}[2] + \frac{1}{4}h_{\text{inv}}[0] = \delta[2] \quad \Rightarrow \quad h_{\text{inv}}[2] = -\frac{1}{4}$

For $n = 3$; $\quad h_{\text{inv}}[3] + \frac{1}{4}h_{\text{inv}}[1] = \delta[3] \quad \Rightarrow \quad h_{\text{inv}}[3] = 0$

For $n = 4$; $\quad h_{\text{inv}}[4] + \frac{1}{4}h_{\text{inv}}[2] = \delta[4] \quad \Rightarrow \quad h_{\text{inv}}[4] = \frac{1}{16}$

For $n = 5$; $\quad h_{\text{inv}}[5] + \frac{1}{4}h_{\text{inv}}[3] = \delta[5] \quad \Rightarrow \quad h_{\text{inv}}[5] = 0$

For $n = 6$; $\quad h_{\text{inv}}[6] + \frac{1}{4}h_{\text{inv}}[4] = \delta[6] \quad \Rightarrow \quad h_{\text{inv}}[6] = -\frac{1}{64}$

$$\vdots$$

$$h_{\text{inv}}[n] = \left(\frac{1}{2}\right)^n \cos\left(\frac{n\pi}{2}\right) u[n]$$

Ch2: LTI Systems

Causal LTI Systems Described By Differential And Difference Equations

- An extremely important class of continuous-time systems is that for which the input and output are related through a ***linear constant-coefficient differential equation (LCCDE)***. Equations of this type arise in the description of a wide variety of systems and physical phenomena.

- Correspondingly, an important class of discrete-time systems is that for which the input and output are related through a ***linear constant-coefficient difference equation (LCCDE)***. Equations of this type are used to describe the sequential behavior of many different processes.

Ch2: LTI Systems

Linear Constant-Coefficient Differential Equations

- Let us consider a first-order differential equation as;

$$\frac{dy(t)}{dt} + 2y(t) = x(t),$$

where *y(t)* denotes the output of the system and *x(t)* is the input.

- The complete solution of eq. above consists of the sum of a *particular solution,* $y_p(t)$, and a *homogeneous solution,* $y_h(t)$, i.e.,

$$y(t) = y_p(t) + y_h(t),$$

- By assuming IRC !!!! The system will be causal and LTI !!!!!

Ch2: LTI Systems

Linear Constant-Coefficient Differential Equations

- The same ideas extend directly to systems described by higher order differential equations. A general **Nth-order** linear constant-coefficient differential equation is given by;

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

- Let us consider N = 0;

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

- Under the condition of initial rest, the system described by eq. above is causal and LTI. !!!!!

Ch2: LTI Systems

Linear Constant-Coefficient Difference Equations

- The discrete-time counterpart of previous eq. is the Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \quad \text{or} \quad y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}.$$

- Let us consider N = 0;

RECURSIVE EQUATION.

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k].$$

NONRECURSIVE EQUATION.

- Under the condition of initial rest, the system described by eq. above is again causal and LTI. !!!!!

Ch2: LTI Systems

Linear Constant-Coefficient Difference Equations

- Look at the equation; $$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k].$$
- That is, eq. above is nothing more than the convolution sum.

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- Note that the impulse response for it has finite duration; that is, it is nonzero only over a finite time interval. Because of this property, the system specified by eq. above is often called *a **finite impulse response (FIR) system***.
- Another type is known as ***infinite impulse response (IIR) system***. For example;

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

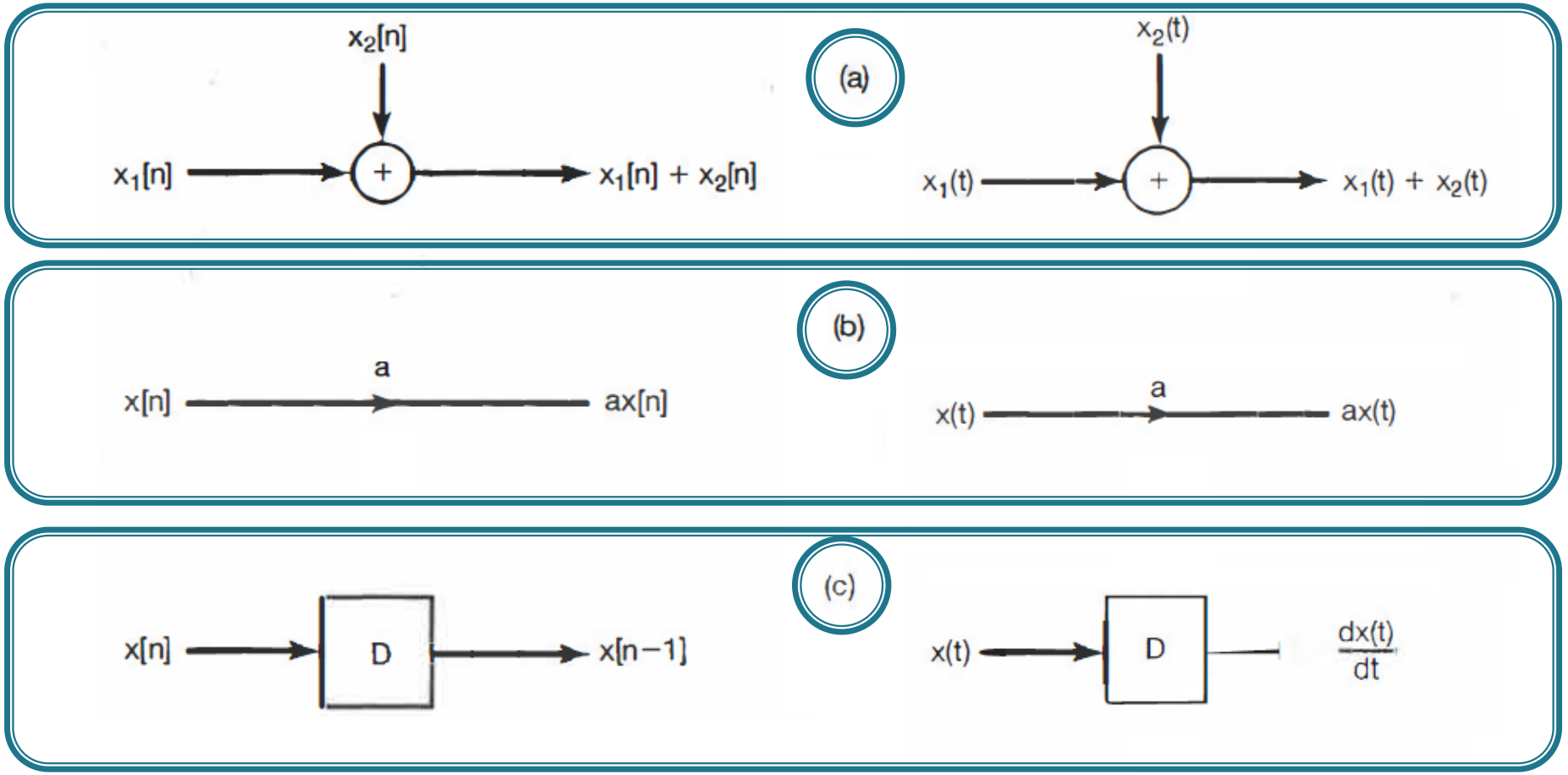
Ch2: LTI Systems

Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

- An important property of systems described by linear constant-coefficient difference and differential equations is that they can be represented in very simple and natural ways in terms of block diagram interconnections of elementary operations. This is significant for a number of reasons.
  - One is that it provides a pictorial representation which can add to our understanding of the behavior and properties of these systems.
  - In addition, such representations can be of considerable value for the simulation or implementation of the systems.
  - In addition, the corresponding representation for discrete-time difference equations suggests simple and efficient ways in which the systems that the equations describe can be implemented in digital hardware.
- Now, we illustrate the basic ideas behind these block diagram representations.

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Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

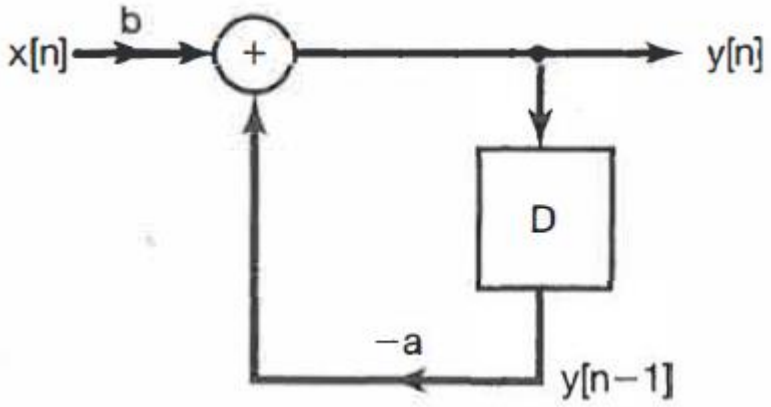


Basic elements for the block diagram representation of the causal system (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

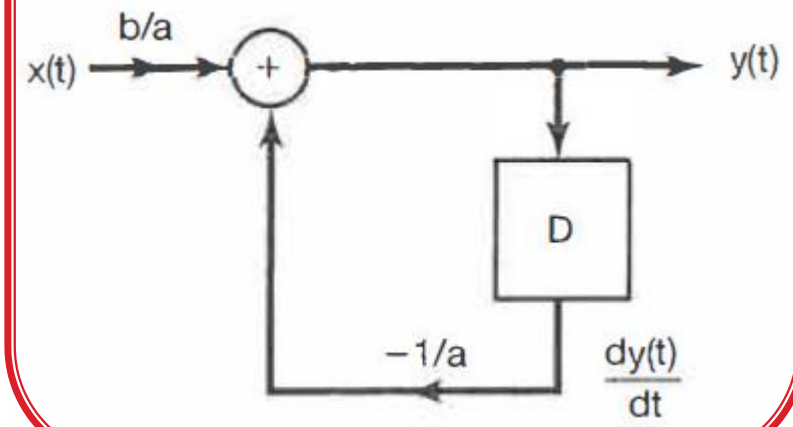
Ch2: LTI Systems

Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

$$y[n] + ay[n-1] = bx[n].$$



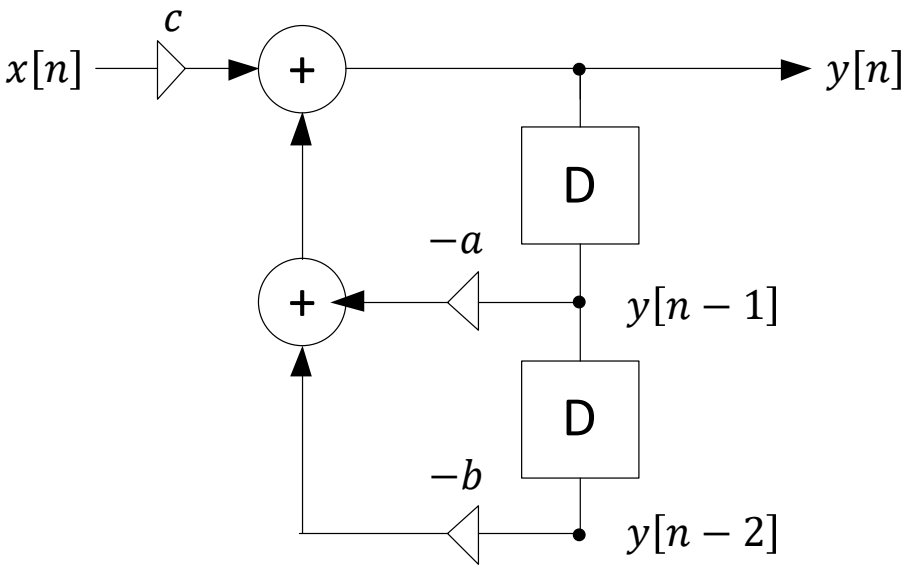
$$\frac{dy(t)}{dt} + ay(t) = bx(t).$$



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Ex:

$$y[n] + ay[n-1] + by[n-2] = cx[n]$$
$$y[n] = -ay[n-1] - by[n-2] + cx[n]$$



END OF CHAPTER 2

Dr. Yılmaz KALKAN